Journal of Frame and Matrix Theory (JFMT) 1 (2023) No. 1, 9–14
Published by Faculty of Mathematics and Computer Science
Hakim Sabzevari University
http://dx.doi.org/10.22034/JFMT.2023.421933.1008

A modification to $L_{p,\alpha}$ and its applicability in error estimation of triangular functions

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(Communicated by Mehdi Zaferanieh)

Abstract

Error estimate and rate of convergence are two important topics in the field of numerical analysis. A convenient normed space corresponding to the problem under regard can have better upper bounds. This paper introduces a weighted normed space $L_{p,\omega}$ which from the measure theory point of view, is a special case of L^p space. This space is a modification of $L_{p,\alpha}$ space, which is introduced before in [2]. Next, by using $L_{p,\alpha}$-norm, we compute a two-variable upper bound of the triangular function.

Keywords: $L_{p,\alpha}$ space; $L_{p,\omega}$ space; Error estimation; Triangular functions.
MSC 2020: Primary: 45D05; Secondary: 65R20, 54H25.

1 Introduction

In 2016, an article entitled "On fractional Langevin equation involving two fractional orders", was published which speaks about the existence and uniqueness of the solution of the two-fractional Langevin equation [2],

$$\begin{cases} \mathcal{D}^{\beta}(\mathcal{D}^{\alpha}+\gamma)x(t)=f(t,x(t)), & 0\le t\le 1, \\ x(0)=\mu_0, \quad x^{(\alpha)}(0)=\nu_0, \end{cases} \tag{1.1}$$

where γ, μ_0, and ν_0 are some given real number, $0<\alpha\le 1$ and $0<\beta\le 1$ have the limitation $0<\alpha+\beta<1$, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a Lebesgue measurable function, and $\mathcal{D}^{\alpha}$ is the notation of the Caputo fractional derivatives of the order α defined by

$$\mathcal{D}^{\alpha}x(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}x^{(m)}(s)ds, \quad 0\le t\le 1. \tag{1.2}$$

The generalized Langevin equation (1.1) has been applied to model physical events in vacillating environments such as modelling the ocean surface wind speed [5].

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Received: 22 October 2023 *Accepted:* 10 November 2023

In that paper, the author has introduced the space $L_{p,\alpha}$, equipped with the associated norm $\|.\|_{p,\alpha}$, for the fixed components $1 \leq p < \infty$ and $0 < \alpha \leq 1$ as follows:

$$L_{p,\alpha}([0,t_f]) := \{\mathrm{f} : \mathrm{f} \text{ is measurable on } [0,\mathrm{t_f}] \text{ and } \|\mathrm{f}\|_{\mathrm{p},\alpha} < \infty\},$$

where

$$\|f\|_{p,\alpha} := \sup_{0\leq t\leq t_f} \Big(\int_0^t \frac{|f(s)|^p}{(t-s)^{1-\alpha}} \mathrm{d}s \Big)^{1/p}.$$

According to the reference [2], some properties of $(L_{p,\alpha}, \|.\|_{p,\alpha})$ are summarized as follows:

(i) $\|f+g\|_{p,\alpha} \leq \|f\|_{p,\alpha} + \|f\|_{p,\alpha}$, which shows that $L_{p,\alpha}$ is a normed space.

(ii) $L_{p,\alpha}$ is a Banach space.

(iii) If $\alpha < \beta$ and $0 \leq t_f \leq 1$, then $\|f\|_{p,\beta} \leq \|f\|_{p,\alpha}$, and clearly $L_{p,\alpha} \subseteq L_{p,\beta}$.

By taking $\beta = 1$ in part (iii) of the above properties, it can be verified that $\|f\|_p \leq \|f\|_{p,\alpha}$. This clearly shows that $L_{p,\alpha} \subseteq L^p$. Recall that $(L^p, \|.\|_p)$ are the set of all functions which the pth power of the absolute value is Lebesgue integrable. It is worth noticing that $L_{p,1}([0,t_f]) = L^p([0,t_f])$.

2 A modification to $L_{p,\alpha}$

In this section, we introduce an improvement of the previous normed spaces, called $L_{p,\omega(\Omega)}$ spaces. As we see, the norm in such spaces is defined in the term of weighted functions. Next, we state the completeness of $L_{p,\omega(\Omega)}$ for $1 \leq p \leq \infty$ that has an essential role in error estimation.

2.1 Weighted spaces $L_{p,\omega}(\Omega)$

Consider $\Omega := \prod_{i=1}^d (a_i, b_i)$ be a Lebesgue measurable subset of $\mathbb{R}^d$, $d \in \mathbb{N}$ with non-empty interior, and $\Omega' := \prod_{i=1}^d (a_i, t_i)$ be a subset of Ω. Let f be a Lebesgue measurable function on Ω and let $\omega(t,s)$ be a given weight function, which is almost everywhere (a.e.) positive on $\Omega \times \Omega'$, and Lebesgue integrable on Ω with the condition $k_\omega := ess\ \sup_\Omega \int_{\Omega'} w(t,s)ds < \infty$. In what follows, we introduce the space $L_{p,\omega}(\Omega)$ and its norm $\|.\|_{L_{p,\omega}(\Omega)}$ for some p, $1 \leq p \leq \infty$, and some useful properties of this space [7].

Definition 2.1. For $1 \leq p \leq \infty$, let

$$L_{p,\omega}(\Omega) := \{f : f \textit{ is Lebesgue measurable on } \Omega \textit{ and } \|f\|_{L_{p,\omega}(\Omega)} < \infty\},$$

where for $1 \leq p < \infty$,

$$\|f\|_{L_{p,\omega}(\Omega)} := ess\ \sup_{t\in\Omega} \Big(\int_{\Omega'} |f(s)|^p w(t,s) ds \Big)^{1/p},$$

and

$$\|f\|_{L_{\infty,\omega}(\Omega)} := ess\ \sup_{s\in\Omega} |f(s)| = \|f\|_{L^\infty(\Omega)}.$$

The space $L_{\infty,\omega}(\Omega)$ consists of all essentially bounded measurable functions f on Ω that there exists a constant K such that $|f(s)| \leq K$ a.e. on Ω. The greatest lower bound of such constants K is called the essential supremum of $|f(s)|$ on Ω, denoted by $\|f\|_{\infty,\omega}(\Omega)$.

The norm $\|.\|_{\infty,\omega}(\Omega)$ is a combination of two classical norms L^p (p-norm) on Ω' with measure $d\mu = \omega(t,s)ds$ and L^∞ (ess sup-norm) on Ω. Hence all norm axioms are evident. For example Minkowski's inequality and Hölder's inequality are easily deduced for $\|.\|_{\infty,\omega}(\Omega)$ from this issue.

It is easily verified that if the weight function ω be a one-variable function of s, and $\Omega = \Omega'$ then $(L_{p,\omega}(\Omega), \|.\|_{L_{p,\omega}(\Omega)})$ is really an extension of the simple weighted space $(L^p_\omega(\Omega), \|.\|_{L^p_\omega(\Omega)})$ [3, 7].

Now we present some of the fundamental properties of $L_{p,\omega}(\Omega)$ spaces.

2.2 Some properties of $L_{p,\omega}(\Omega)$

Lemma 2.2. Let $1 \le p < \infty$. If $\omega(t,s) \ge 1$ for all $(t,s) \in \Omega \times \Omega'$ (a.e.), then $\|f\|_{L^p(\Omega)} \le \|f\|_{L_{p,\omega}(\Omega)}$, and clearly $L_{p,\omega}(\Omega) \subseteq L^p(\Omega)$. Otherwise, $\|f\|_{L_{p,\omega}(\Omega)} \le \|f\|_{L^p(\Omega)}$, and therefore $L_{p,\omega}(\Omega) \subseteq L^p(\Omega)$.

Proof. The proof deduces immediately from definition of the spaces $L_{p,\omega}(\Omega)$ and $L^p(\Omega)$.

□

Lemma 2.3. Suppose $1 \le p < \infty$. If $\omega_1(t,s) \le \omega_2(t,s)$ for all $(t,s) \in \Omega \times \Omega'$ (a.e.), then $\|f\|_{L_{p,\omega_1}(\Omega)} \le \|f\|_{L_{p,\omega_2}(\Omega)}$, and clearly $L_{p,\omega_2}(\Omega) \subseteq L_{p,\omega_1}(\Omega)$.

Proof. It follows immediately from the following obvious inequality,

$$\int_{\Omega'} |f(s)|^p \omega_1(t,s)ds \le \int_{\Omega'} |f(s)|^p \omega_2(t,s)ds.$$

□

The following theorem establishes the completeness of $L_{p,\omega}(\Omega)$ for $1 \le p \le \infty$.

Theorem 2.4. For $1 \le p \le \infty$, $L_{p,\omega}(\Omega)$ is a complete metric space.

Proof. The proof is similar to the proof of completeness of L^p (see [6], Theorem 3.11, p. 67).

□

Remark 2.5. In Definition 2.1, if we choose $\omega(t,s) := \frac{1}{(t-s)^{1-\alpha}}$ for $0 < \alpha \le 1$, and $\Omega := [0,1]$, then we obtain the space $(L_{p,\alpha}[0,1], \|.\|_{p,\alpha})$. Special structure of this space has made it perfect for the fractional integral equations. With this agreement, the $L_{p,\alpha}$-norm, for an arbitrary function f, can be written as the Riemann-Liouville fractional integral operator, i.e.,

$$\|f\|_{p,\alpha} = \sup_{0 \le t \le 1} \Big(\Gamma(\alpha) I^\alpha |f|^p(t)\Big)^{1/p},$$

where I^α is the Riemann-Liouville operator of the order α, which is defined as follows:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s.$$

3 Application of the space $L_{p,\alpha}$ in error estimation

In this part we offer some reasons for applicability of the normed space $(L_{p,\alpha}, \|.\|_{p,\alpha})$ as follows.

(1) $L_{p,\alpha}[0,1]$-norm is more flexible than the classical norm $L^p[0,1]$, because there is an additional component α in its structure. Also, we can easily verify that $\lim_{\alpha\to 1} \|.\|_{p,\alpha} = \|.\|_p$. This shows that the $L_{p,\alpha}[0,1]$ space is a generalization of $L^p[0,1]$ space.

(2) Using the normed space $(L_{p,\alpha}, \|.\|_{p,\alpha})$, we can obtain some better error bounds in approximation theory. For detailed discussion, consider the following triangular functions over the subinterval $[ih,(i+1)h]$, $i = 0,1,\dots,m-1$ of the interval $[0,t_f]$,

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h}, & ih \le t \le (i+1)h, \\ 0, & \text{elsewhere}, \end{cases} \qquad T2_i(t) = \begin{cases} \frac{t-ih}{h}, & ih \le t \le (i+1)h, \\ 0, & \text{elsewhere}, \end{cases} \tag{3.1}$$

where for a given positive integer m, the step size h is defined as $h = \frac{t_f}{m}$ (see [2]). Any square integrable function f can be approximated by using two m-set vectors

$$T1(t) = [T1_0(t), \dots, T1_{m-1}(t)]^T, \qquad T2(t) = [T2_0(t), \dots, T2_{m-1}(t)]^T,$$

as follows:

$$f(t) \simeq \widetilde{f}(t) = F1^T T1(t) + F2^T T2(t),$$

where $F1_i = f(ih)$, and $F2_i = f((i+1)h)$, $i = 0, 1, \dots, m-1$. An error bound for estimating a twice differentiable function f by the triangular orthogonal basis functions denoted by f_m, is derived in $L_{p,\alpha}$-norm as follows: Let $f_m^i(t)$ be the TF estimation of the function $f(t)$ in the subinterval $[ih, (i+1)h]$, which has the following form:

$$\begin{aligned} f_m^i(t) &= f(ih)T1_i(t) + f((i+1)h)T2_i(t) \\ &= f(ih)\Big(1 - (\frac{t-ih}{h})\Big) + f((i+1)h)\Big(\frac{t-ih}{h}\Big) \\ &= f(ih) + f((i+1)h)\Big(\frac{t-ih}{h}\Big) - f(ih)\Big(\frac{t-ih}{h}\Big) \\ &= f(ih) + \Big(\frac{f((i+1)h) - f(ih)}{h}\Big)(t-ih) \\ &\simeq f(ih) + f'(ih)(t-ih). \end{aligned}$$

Defining the function f^i as

$$f^i(t) = \begin{cases} f(t), & ih \le t < (i+1)h, \\ 0, & \text{elsewhere}, \end{cases} \tag{3.2}$$

and then expanding it by the second order Taylor series near the center point ih, there exists a number $\xi_t^i \in (ih, (i+1)h)$ such that

$$f^i(t) = f(ih) + f'(ih)(t-ih) + \frac{f''(\xi_t^i)}{2!}(t-ih)^2.$$

We now define the interval error function $e_i(t)$ for all $t \in [ih, (i+1)h)$ as

$$e_i(t) = \begin{cases} f^i(t) - f_m^i(t), & ih \le t < (i+1)h, \\ 0, & \text{elsewhere}. \end{cases} \tag{3.3}$$

Assuming f is twice continuously differentiable on $[0, t_f]$, there exists a positive constant M such that $|f''(t)| \le M$, for all $t \in [0, t_f]$. On the other hand, the Riemann-Liouville operator of the power function $(t-a)^{\beta-1}$, for $\alpha, \beta > 0$ yields [4]:

$$I^\alpha (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t-a)^{\alpha+\beta-1}, \quad a \le t.$$

Therefore for some $\xi_t^i \in (ih, (i+1)h)$, $L_{p,\alpha}$-norm of e_i in the i-th interval is as follows:

$$\begin{aligned} \|e_i\|_{p,\alpha}^p = \sup_{0 \le t \le t_f} \Big(\int_0^t \frac{|e_i(s)|^p}{(t-s)^{1-\alpha}} \mathrm{d}s\Big) &= \sup_{ih \le t \le (i+1)h} \Big(\int_{ih}^t \frac{|f^i(s) - f_m^i(s)|^p}{(t-s)^{1-\alpha}} \mathrm{d}s\Big) \\ &= \sup_{ih \le t \le (i+1)h} \Big(\int_{ih}^t \frac{1}{(t-s)^{1-\alpha}} \Big|\frac{f''(\xi_s^i)}{2!}(s-ih)^2\Big|^p \mathrm{d}s\Big) \\ &\le \frac{M^p}{2^p} \sup_{ih \le t \le (i+1)h} \Big(\int_{ih}^t \frac{(s-ih)^{2p}}{(t-s)^{1-\alpha}} \mathrm{d}s\Big) \\ &\le \frac{M^p}{2^p} \frac{\Gamma(\alpha)\Gamma(2p+1)}{\Gamma(\alpha+2p+1)} \sup_{ih \le t \le (i+1)h} \Big((t-ih)^{2p+\alpha}\Big) \\ &\le \frac{M^p}{2^p} \frac{\Gamma(\alpha)\Gamma(2p+1)}{\Gamma(\alpha+2p+1)} h^{2p+\alpha}. \end{aligned} \tag{3.4}$$

This states that for any $i \in \{0, 1, \dots, m-1\}$, we have

$$\|e_i\|_{p,\alpha} \le \frac{M}{2}\Big(\frac{\Gamma(\alpha)\Gamma(2p+1)}{\Gamma(\alpha+2p+1)}\Big)^{1/p} h^{(2p+\alpha)/p}. \tag{3.5}$$

We see clearly that for any $t \in [0, t_f]$, there exists $i \in \{0, 1, \dots, m-1\}$ such that t belongs to the interval $[ih, (i+1)h]$. Therefore,

$$|f(t) - f_m(t)| \le \max_{i \in \{0,1,\dots,m-1\}} |f^i(t) - f_m^i(t)|.$$

So,

$$
\begin{aligned}
\|e\|_{p,\alpha}^p &= \sup_{0\le t\le t_f}\left(\int_0^t \frac{|f(s)-f_m(s)|^p}{(t-s)^{1-\alpha}}\mathrm{d}s\right)\\
&\le \sup_{0\le t\le t_f}\left(\int_0^t \frac{\max_{i\in\{0,1,\ldots,m-1\}}|f^i(s)-f_m^i(s)|^p}{(t-s)^{1-\alpha}}\mathrm{d}s\right)\\
&\le \max_{i\in\{0,1,\ldots,m-1\}}\sup_{ih\le t\le (i+1)h}\left(\int_{ih}^t \frac{|f^i(s)-f_m^i(s)|^p}{(t-s)^{1-\alpha}}\mathrm{d}s\right)\\
&= \frac{\Gamma(\alpha)\Gamma(2p+1)}{\Gamma(\alpha+2p+1)}\,\frac{M^p}{2^p}h^{2p+\alpha}\\
&= \frac{\Gamma(\alpha)\Gamma(2p+1)}{\Gamma(\alpha+2p+1)}\,\frac{M^p t_f^{2p+\alpha}}{2^p m^{2p+\alpha}}.
\end{aligned}
\tag{3.6}
$$

This shows that $\|e\|_{p,\alpha} \le \left(\frac{B_{p,\alpha}}{m^{2p+\alpha}}\right)^{1/p}$, where the constant $B_{p,\alpha} := \frac{\Gamma(\alpha)\Gamma(2p+1)}{\Gamma(\alpha+2p+1)}\,\frac{M^p t_f^{2p+\alpha}}{2^p}$ depends only on p and α. Recall that M is the positive constant such that $|f''(t)| \le M$, for all $t \in [0, t_f]$. As can be seen from (3.6), the upper bound $\epsilon_m(p,\alpha) := \left(\frac{B_{p,\alpha}}{m^{2p+\alpha}}\right)^{1/p}$ is a two variable function of the elements $1 \le p \le \infty$ and $0 < \alpha \le 1$. Without the norm $\|.\|_{p,\alpha}$ we can only compute the error bound $\epsilon_m(p,1)$ which is deduced from $\lim_{\alpha\to 1}\|.\|_{p,\alpha} = \|.\|_p$. Applying this error bound, we able to give a better convergence analysis of the triangular orthogonal bases. For example we can obtain the optimum upper bound of $\epsilon_m(p,\alpha)$ in terms of the variables p and α. For this issue, we have used the *optimization* package of *Maple* software 2015. After this work, the optimal values of $1 \le p \le \infty$ and $0 \le \alpha \le 1$ are computed as $p \simeq 6.831832$ and $\alpha = 1$. This shows that we don't always do any error analysis only with L^2-norm. The behaviour of the upper bound $\epsilon_m(p,\alpha)$ assuming $M = 1$, $t_f = 1$ and $m = 5$ is depicted for various tolerances of p and α in Fig. 1.

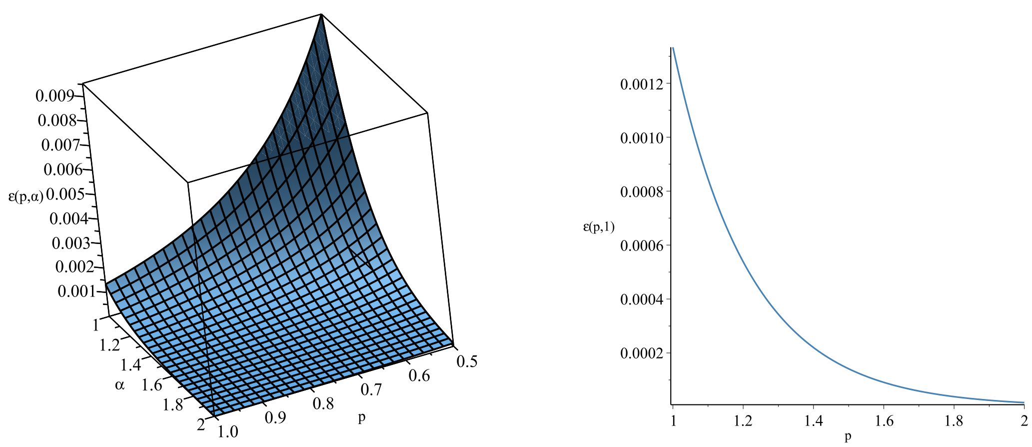

Figure 1: (Left) the error bound $\epsilon_m(p,\alpha)$ under $\|.\|_{p,\alpha}$-norm for $1 \le p \le 2$ and $0.5 \le \alpha \le 1$ and (right) the error bound $\epsilon_m(p,1)$ under $\|.\|_p$-norm for $1 \le p \le 2$.

4 Conclusion

In this paper, we introduced a modified normed space of $L_{p,\alpha}$, named $L_{p,\omega}$-space. This space is a special case of L^p which is obtained by defining the measure function $d\mu := \omega(t,s)ds$. Thus, $(L_{p,\omega}(\Omega), \|.\|_{L_{p,\omega}(\Omega)})$ is a Banach space. Then, we obtained the two-variable upper bound function $\epsilon_m(p,\alpha)$ for the triangular functions in $L_{p,\alpha}$-space. Applying the *optimization* package of *Maple* software 2015, the optimal values of $1 \le p \le \infty$ and $0 < \alpha \le 1$ with fixed parameters $M = 1$, $t_f = 1$ and $m = 5$ were computed as $p \simeq 6.831832$ and $\alpha = 1$, respectively. It is worth noticing that the optimal value of α in $\epsilon_m(p,\alpha)$, for any $1 \le p \le \infty$ and $0 \le \alpha \le 1$, is one, because $\|f\|_p \le \|f\|_{p,\alpha}$.

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