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On c-completely regular frames

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Abstract

Motivated by definitions of countable completely regular spaces and completely below relations of frames, we define what we call a c-completely below relation, denoted by $\prec\!\!\prec_c$, in between two elements of a frame. We show that $a \prec\!\!\prec_c b$ for two elements a, b of a frame L if and only if there is $\alpha \in \mathcal{R}L$ such that $\operatorname{coz}(\alpha) \wedge a = 0$ and $\operatorname{coz}(\alpha - \mathbf{1}) \leq b$ where the set $\{r \in \mathbb{R} : \operatorname{coz}(\alpha - \mathbf{r}) \neq 1\}$ is countable. We say a frame L is a c-completely regular frame if $a = \bigvee_{x \prec\!\prec_c a} x$ for any $a \in L$. It is shown that a frame L is a c-completely regular frame if and only if it is a zero-dimensional frame. An ideal I of a frame L is said to be c-completely regular if $a \in I$ implies $a \prec\!\!\prec_c b$ for some $b \in I$. The set of all c-completely regular ideals of a frame L, denoted by $\mathrm{c-CRegId}(L)$, is a compact regular frame and it is a compactification for L whenever *it* is a c-completely regular frame. We denote this compactification by $\beta_c L$ and it is isomorphic to the frame $\beta_0 L$, that is, Stone-Banaschewski compactification of L. Finally, we show that open and closed quotients of a c-completely regular frame are c-completely regular.

Keywords: Frame; c-completely regular frame and space; c-completely below relation; c-completely regular ideals; Zero-dimensional frame; Compactification of frame.
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1 Introduction

As usual, let $C(X)$ be the ring of all continuous real-valued functions on a completely regular space X. In [12], the authors introduced and studied the subalgebra $C_c(X)$ of $C(X)$ consisting of elements with a countable range. In that paper, a Hausdorff space X is called countable completely regular (briefly, c-completely regular) if whenever $F \subseteq X$ is a closed set and $x \notin F$, then there exists $f \in C_c(X)$ with $f(F) = 0$ and $f(x) = 1$. Equivalently, a Hausdorff space X is c-completely regular if whenever $F \subseteq X$ is a closed set and $x \notin F$, then there exist $g, h \in C_c(X)$ with $x \in X \setminus Z(h) \subseteq Z(g) \subseteq X \setminus F$. Therefore, a Hausdorff space X is c-completely regular if and only if for any open set V of X, there is $\{(U_i, f_i)\}_{i\in I} \subseteq \mathfrak{O}(X) \times C_c(X)$ such that $U_i \cap \operatorname{coz}(f_i) = \emptyset$, $\operatorname{coz}(f_i - \mathbf{1}) \subseteq V$ and $V = \bigcup_{i\in I} U_i$. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}(X)$.

A *frame* is a complete lattice L in which

$$a \wedge \bigvee S = \bigvee_{s\in S} a \wedge s$$

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for any $a \in L$ and $S \subseteq L$, $\wedge$ and $\bigvee$ implicating meet and join in L, as usual. We use 0 and 1 for the *bottom element* and the *top element* of L, respectively. Let $\mathcal{R}L$ be the ring of all continuous real-valued functions on a completely regular frame L (see [2, 3] for details). For any $\alpha \in \mathcal{R}L$, let $R_\alpha = \{r \in \mathbb{R} : \mathrm{coz}(\alpha - \mathbf{r}) \neq 1\}$ (see [9]). The authors in [10, 11, 15] study the set $\mathcal{R}_c(L) = \{\alpha \in \mathcal{R}L : R_\alpha$ is countable$\}$ as a sub-f-ring of $\mathcal{R}L$ (also, see [6, 7, 8]). When we study the ring $\mathcal{R}_c(L)$, we can assume that L is a zero-dimensional (c-completely regular) frame because, in [15], it is shown that for any frame L there exists a zero-dimensional frame M which is a continuous image of L and $\mathcal{R}_c(L) \cong \mathcal{R}_c(M)$. In [10], it is shown that $\mathrm{Coz}_c L = \{\mathrm{coz}(\alpha) : \alpha \in \mathcal{R}_c(L)\}$ is a sub-σ-frame of L.

As usual, the *rather below* and the *completely below* relations denoted by $\prec$ and $\prec\!\!\prec$, respectively. Let L be a frame. Recall that $a \prec b$ in case there is an element $c \in L$ such that $a \wedge c = 0$ and $c \vee b = 1$. Motivated by this definition, in Definition 3.1, a c-rather below relation of L is defined. Some lattice-theoretic properties of this relation are given in Theorem 3.2. Recall that $a \prec\!\!\prec b$ in case there are elements (x_q) indexed by the rational numbers $[0, 1] \cap \mathbb{Q}$ such that $a = x_0$, $b = x_1$ and $x_p \prec x_q$ for $p < q$. In Definition 3.5, we define a c-completely below relation of L and we summarize the lattice-theoretic properties of this relation in Theorem 3.6. We show that the relations of $\prec$, $\prec\!\!\prec$, $\prec_c$ and $\prec\!\!\prec_c$ are equal for compact c-regular frames (Corollary 4.3)

The main interest of the completely below relation, however, lies in its connection with continuous $\mathcal{L}(\mathbb{R})$-valued maps, as indicated by the following theorem (see [2, Poroposition 2.1.4] and [14, IV 1.4]).

Theorem 1.1. The following are equivalent for any $a, b \in L$.

(1) $a \prec\!\!\prec b$.

(2) There is $\alpha \in \mathcal{R}L$ such that $\mathrm{coz}(\alpha) \wedge a = 0$ and $\mathrm{coz}(\alpha - \mathbf{1}) \leq b$. Such a map can be chosen to satisfy $\mathbf{0} \leq \alpha \leq \mathbf{1}$ when it exists.

(3) There is some $c \in \mathrm{Coz}L$ such that $a \prec c \prec b$.

This theorem is extended to the c-completely below relation in Theorem 3.9.

We define and study c-completely regular frames in the last section. For any frame L, the assignment $a \mapsto \{x \in L : x \prec\!\!\prec_c a\}$ defines a map $r_c : L \to \mathrm{c-CRegId}(L)$ such that r_c is a right adjoint to $\bigvee$. Some properties of this map are given in Lemma 5.5. We show that $\mathrm{c-CRegId}(L)$ is a compact regular frame and $\bigvee : \mathrm{c-CRegId}(L) \to L$ is a compactification of L if it is a c-completely regular frame (Lemma 5.6). In Theorem 5.8, it is shown that for any frame L, $\mathrm{c-CRegId}(L)$ is a c-completely regular frame if and only if $t \prec\!\!\prec_c a$ implies that $r_c(t) \prec\!\!\prec_c r_c(a)$ in $\mathrm{c-CRegId}(L)$ for any $a \in L$ and $t \in \mathrm{Coz}_c L$. Finally we show that any frame map preserves $\prec_c$ and $\prec\!\!\prec_c$, and hence, any homomorphic image of a c-completely regular frame is a c-completely regular frame (see Lemma 5.9).

2 Preliminaries

2.1 Frames

For a general theory of frames and locales, we refer to [13, 14]. A *frame* or *locale* L is a complete lattice in which finite meets distribute over arbitrary joins.

A frame L is said to be *compact* if whenever $1 = \bigvee S$, for $S \subseteq L$, then $1 = \bigvee T$ for some finite subset $T \subseteq S$. A *frame homomorphism* (or *frame map*) is a map between frames which preserves finite meets and arbitrary joins. Frame homomorphisms which are onto will frequently be referred to as *quotient maps*. In particular, for any $a \in L$, the *open and closed quotients* are defined by $\downarrow a = \{x \in L : x \leq a\}$ and $\uparrow a = \{x \in L : a \leq x\}$, respectively. A homomorphism is called *dense* if it maps only the bottom element to the bottom element. A *compactification* of L is a dense onto homomorphism $h : M \to L$ with compact regular domain.

An *ideal*, in any bounded distributive lattice A, is a subset $I \subseteq A$ such that $\bigvee J \in I$ for any finite $J \subseteq I$, and $x \in I$ whenever $x \leq y$ and $y \in I$. The set $Id(A)$ of all ideals of A is a frame, with $\leq$ as inclusion and the ideal generated by $\bigcup I_\alpha$ as $\bigvee I_\alpha$, and $\bigwedge I_\alpha = \bigcap I_\alpha$. Also, for every $I, J \in Id(A)$

1. $I \wedge J = I \cap J = \{a \wedge b : a \in I, b \in J\}$.
2. $I \vee J = \{a \vee b : a \in I,\, b \in J\}$.

The *pseudocomplement* of an element a in a frame L, denoted by a^*, is the element

$$a^* = \bigvee\{x \in L : a \wedge x = 0\}.$$

An element $a \in L$ is called *complemented* if $a \vee a^* = 1$. We write

$$BL = \{a \in L : a \vee a^* = 1\}$$

for the set of all complemented elements of L and, clearly, it is a sublattice of L.

2.2 The ring $\mathcal{R}_c(L)$

The ring $\mathcal{R}_c(L) = \{\alpha \in \mathcal{R}L : R_\alpha \text{ is countable}\}$, where $R_\alpha = \{r \in \mathbb{R} : \mathrm{coz}(\alpha - \mathbf{r}) \neq 1\}$, has been studied as a sub-f-ring of $\mathcal{R}L$ (see [10, 15] for details).

Recall that a σ-frame is a bounded distributive lattice in which every countable subset has a join and binary meet distributes over these joins, and regularity (complete regularity) of a σ-frame is the countable counterparts of regularity (complete regularity) of frames, that is, $a = \bigvee_{a_n \prec a} a_n$ ($a = \bigvee_{a_n \prec\!\!\prec a} a_n$) for each element a.

In [10], it is shown that $\mathrm{Coz}_c L = \{\mathrm{coz}(\alpha) : \alpha \in \mathcal{R}_c(L)\}$ is a sub-σ-frame of L such that

$$s \in \mathrm{Coz}_c L \Leftrightarrow s = \bigvee_{n=1}^{\infty} s_n, \text{ where } s_n \in BL.$$

This is to say that $\mathrm{Coz}_c L$ is a regular sub-σ-frame of L and hence, by [5], we deduce that it is normal (that is, given a an b with $a \vee b = 1$, we can find c and d such that $c \wedge d = 0$ and $a \vee c = 1 = b \vee d$). So, in $\mathrm{Coz}_c L$, we have $\prec = \prec\!\!\prec$. We note that $BL \subseteq \mathrm{Coz}_c L$ for any frame L.

3 c-completely below relation

Recall that $a \prec b$ in case there is an element $c \in L$ such that $a \wedge c = 0$ and $c \vee b = 1$. This motivates the following definition.

Definition 3.1. Let L be a frame and $a, b \in L$. We define the order $\prec_c$ on L by

$$a \prec_c b \Leftrightarrow \text{ there exists } x \in \mathrm{Coz}_c L \text{ such that } a \wedge x = 0 \text{ and } x \vee b = 1.$$

If $a \prec_c b$ we say that a is c-rather below b.

We note that $0 \prec_c a$ and $a \prec_c 1$ for any $a \in L$. It is clear that if $a, b \in \mathrm{Coz}_c L$, then $a \prec_c b$ if and only if $a \prec b$ in $\mathrm{Coz}_c L$. We collect some lattice-theoretic properties of the c-rather below relation in the next theorem.

Theorem 3.2. Let L be a frame and $a, b, c, d \in L$.

(1) If $a \prec_c b$, then $a \prec b$.
(2) If $a \prec_c b$, then there exists $x \in \mathrm{Coz}_c L$ such that $a \leq x^* \prec b$.
(3) $a \prec_c a$ if and only if a is complemented.
(4) If $a \leq c \prec_c d \leq b$, then $a \prec_c b$.
(5) If $a \leq b$ and b is complemented, then $a \prec_c b$.
(6) If $a \prec_c b$ and $c \prec_c d$, then $a \vee c \prec_c b \vee d$ and $a \wedge c \prec_c b \wedge d$.
(7) $a \vee b \prec_c c \Leftrightarrow a \prec_c c, b \prec_c c$.
(8) $c \prec_c a \wedge b \Leftrightarrow c \prec_c a, c \prec_c b$.

Proof . The proof of (1) is obvious.

(2) Let $a, b \in L$ with $a \prec_c b$ be given. Then there exits $x \in \mathrm{Coz}_c L$ such that $a \wedge x = 0$ and $x \vee b = 1$. The latter implies that

$$1 = x \vee b \leq x^{**} \vee b.$$

This means that $x^* \prec b$. By the former case, we have $a \leq x^*$. Thus $a \leq x^* \prec b$ with $x \in \mathrm{Coz}_c L$.

(3) We have $a \in BL$ if and only if $a^* \vee a = 1$ and $a^* \wedge a = 0$. On the other hand, if $a \in BL$, then $a, a^* \in \mathrm{Coz}_c L$. Thus it is clear that $a \prec_c a$ if and only if a is complemented.

(4) Since $c \prec_c d$, then there exits $x \in \mathrm{Coz}_c L$ such that $c \wedge x = 0$ and $x \vee d = 1$. But $a \leq c$ and $d \leq b$ imply that $a \wedge x \leq c \wedge x = 0$ and $1 = x \vee d \leq x \vee b$. Hence $a \wedge x = 0$ and $x \vee b = 1$ with $x \in \mathrm{Coz}_c L$. This means that $a \prec_c b$.

(5) Let $a \leq b$ and b be complemented. Then by (3), $b \prec_c b$. So by (4), $a \prec_c b$.

(6) Let $a \prec_c b$ and $c \prec_c d$. Then there exist $x, y \in \mathrm{Coz}_c L$ such that $a \wedge x = 0$, $c \wedge y = 0$, $b \vee x = 1$, and $d \vee y = 1$. Now,

$$(a \vee c) \wedge (x \wedge y) = (a \wedge x \wedge y) \vee (c \wedge x \wedge y) = 0 \vee 0 = 0$$

and

$$(b \vee d) \vee (x \wedge y) = (b \vee d \vee x) \wedge (b \vee d \vee y) = 1 \wedge 1 = 1$$

imply that $a \vee c \prec_c b \vee d$ since $x \wedge y \in \mathrm{Coz}_c L$. Also,

$$(a \wedge c) \wedge (x \vee y) = (a \wedge c \wedge x) \vee (a \wedge c \wedge y) = 0 \vee 0 = 0$$

and

$$(b \wedge d) \vee (x \vee y) = (b \vee x \vee y) \wedge (d \vee x \vee y) = 1 \wedge 1 = 1$$

imply that $a \wedge c \prec_c b \wedge d$ since $x \vee y \in \mathrm{Coz}_c L$.

(7) Since $a, b \leq a \vee b$, by (4), we get that if $a \vee b \prec_c c$ then $a \prec_c c$ and $b \prec_c c$. The converse follows by (6).

(8) Since $a \wedge b \leq a, b$, by (4), we get that if $c \prec_c a \wedge b$ then $c \prec_c a$ and $c \prec_c b$. The converse follows by (6). □

Corollary 3.3. Let L be a frame, $a \in L$ and T be a finite subset of L. Then

(1) $\bigvee T \prec_c a \Leftrightarrow t \prec_c a$, for all $t \in T$.

(2) $a \prec_c \bigwedge T \Leftrightarrow a \prec_c t$, for all $t \in T$.

In a frame L, a *scale* from a to b is a subset

$$\{x_q : q \in [0,1] \cap \mathbb{Q}\} \subseteq L,$$

indexed by the rational interval $[0,1] \cap \mathbb{Q}$ such that $a = x_0$, $b = x_1$, and $x_p \prec x_q$ whenever $p < q$ in $[0,1] \cap \mathbb{Q}$.

Definition 3.4. Let a and b be two elements of a frame L. A c-scale from a to b is a subset

$$\{x_q : q \in [0,1] \cap \mathbb{Q}\} \subseteq L,$$

indexed by the rational interval $[0,1] \cap \mathbb{Q}$ such that $a = x_0$, $b = x_1$, $\{x_q : q \in (0,1) \cap \mathbb{Q}\} \subseteq \mathrm{Coz}_c L$ and $x_p \prec_c x_q$ whenever $p < q$ in $[0,1] \cap \mathbb{Q}$.

In the following definition, as stated in the abstract, we shall use definitions of countable completely regular spaces and completely below relations of a frame L to define a c-completely below relation of L which will be the subject of study in this paper.

Definition 3.5. Let L be a frame and $a, b \in L$. We say that a is c-completely below b, and write $a \prec\!\!\prec_c b$, if there is a c-scale from a to b.

Clearly, $0 \prec\!\!\prec_c a$ and $a \prec\!\!\prec_c 1$ for any $a \in L$. Clearly, $a \prec\!\!\prec_c b$ implies $a \prec\!\!\prec b$. The following theorem gives some lattice-theoretic properties of the c-completely below relation.

Theorem 3.6. Let L be a frame and $a, b, c, d \in L$.

(1) If $a \prec\!\!\prec_c b$, then $a \prec_c b$.
(2) $a \prec\!\!\prec_c a$ if and only if a is complemented.
(3) If $a \leq c \prec\!\!\prec_c d \leq c$, then $a \prec\!\!\prec_c b$.
(4) If $a \prec\!\!\prec_c b$ and $c \prec\!\!\prec_c d$, then $b \vee d \prec\!\!\prec_c a \vee c$ and $b \wedge d \prec\!\!\prec_c a \wedge c$.
(5) If $a \prec\!\!\prec_c b$ then there exists $s \in \mathrm{Coz}_c L$ with $a \prec\!\!\prec_c s \prec\!\!\prec_c b$.
(6) $a \vee b \prec\!\!\prec_c c \Leftrightarrow a \prec\!\!\prec_c c, b \prec\!\!\prec_c c$.
(7) $c \prec\!\!\prec_c a \wedge b \Leftrightarrow c \prec\!\!\prec_c a, c \prec\!\!\prec_c b$.

(8) The set $\{x \in L : x \prec\!\!\prec_c a\}$ is an ideal of L.

Proof . The proof of (1), (2) and (3) is clear.

(4). Let $a \prec\!\!\prec_c b$ and $c \prec\!\!\prec_c d$. Let $\{x_q : q \in [0,1] \cap \mathbb{Q}\}$ be a c-scale from a to b, and $\{y_q : q \in [0,1] \cap \mathbb{Q}\}$ be a c-scale from c to d. Then Theorem 3.2 (6) shows that $\{x_q \vee y_q : q \in [0,1] \cap \mathbb{Q}\}$ be a c-scale from $a \vee b$ to $c \vee d$, and $\{x_q \wedge y_q : q \in [0,1] \cap \mathbb{Q}\}$ be a c-scale from $a \wedge b$ to $c \wedge d$. Hence $b \vee d \prec\!\!\prec_c a \vee c$ and $b \wedge d \prec\!\!\prec_c a \wedge c$.

(5). Let $\{x_q : q \in [0,1] \cap \mathbb{Q}\}$ be a c-scale between a and b. Then we can take $x_{\frac{1}{2}}$ to be s, and use the c-scales $\{x_{\frac{q}{2}} : q \in [0,1] \cap \mathbb{Q}\}$ and $\{x_{\frac{q+1}{2}} : q \in [0,1] \cap \mathbb{Q}\}$ to show $a \prec\!\!\prec_c s$ and $s \prec\!\!\prec_c b$, respectively.

(6). By (3) and (4) is obvious.

(7). By (3) and (4) is obvious.

(8). By (6) is obvious. □

An immediate corollary to the foregoing lemma is the following.

Corollary 3.7. Let L be a frame and $a, b \in L$. Then $a \prec\!\!\prec_c b$ if and only if there exists $s \in \mathrm{Coz}_c L$ such that $a \prec\!\!\prec_c s \prec\!\!\prec_c b$.

The main interest of the $\prec\!\!\prec_c$ relation, however, lies in its connection with continuous $\mathcal{L}(\mathbb{R})$-valued maps, as indicated by the following theorem. We begin with the next lemma. This lemma was also proved in [1] although it was stated slightly differently there. Here we state it in a manner we shall find useful.

Lemma 3.8. Suppose $\mathrm{coz}(\varphi) \prec_c \mathrm{coz}(\delta)$.

(1) If $\varphi, \delta \in \mathcal{R}L$, then there exists an invertible element $\rho \in \mathcal{R}L$ such that $\varphi = \varphi\rho\delta^2$.
(2) If $\varphi \in \mathcal{R}L$ and $\delta \in \mathcal{R}_c(L)$, then there exists an invertible element $\rho \in \mathcal{R}_c(L)$ such that $\varphi = \varphi\rho\delta^2$.

Proof . (1). Since $\mathrm{coz}(\varphi) \prec_c \mathrm{coz}(\delta)$, we can find $\alpha \in \mathcal{R}_c(L)$ such that $\mathrm{coz}(\varphi) \wedge \mathrm{coz}(\alpha) = 0$ and $\mathrm{coz}(\alpha) \vee \mathrm{coz}(\delta) = 1$. The latter implies that

$$1 = \mathrm{coz}(\alpha) \vee \mathrm{coz}(\delta) = \mathrm{coz}(\alpha^2) \vee \mathrm{coz}(\delta^2) = \mathrm{coz}(\alpha^2 + \delta^2),$$

this means that $\alpha^2 + \delta^2$ is invertible. By the former case, we have $\mathrm{coz}(\varphi\alpha) = 0$, that is, $\varphi\alpha = \mathbf{0}$. Putting $\rho = \frac{\mathbf{1}}{\alpha^2+\delta^2}$, we then have

$$\varphi = \varphi\frac{\alpha^2 + \delta^2}{\alpha^2 + \delta^2} = \frac{\varphi\delta^2}{\alpha^2 + \delta^2} = \varphi\rho\delta^2.$$

(2). Similar to (1). □

Recall from [3, Lemma 6] that for any $\alpha \in \mathcal{R}L$ and any $p, q \in \mathbb{Q}$,

$$\alpha(p, q) = \mathrm{coz}\big((\alpha - \mathbf{p})^+ \wedge (\mathbf{q} - \alpha)^+\big).$$

So, $\alpha(p, q) \in \mathrm{Coz}_c L$ whenever $\alpha \in \mathcal{R}_c(L)$. We shall use this fact in part of the proof below.

Theorem 3.9. The following are equivalent for any $a, b \in L$.

(1) $a \prec\!\!\prec_c b$.
(2) There are $c, d \in \mathrm{Coz}_c L$ such that $a \le c \prec_c d \le b$.
(3) There are $c \in \mathrm{Coz}L$ and $d \in \mathrm{Coz}_c L$ such that $a \le c \prec_c d \le b$.
(4) There is $\alpha \in \mathcal{R}_c(L)$ such that $\mathrm{coz}(\alpha) \wedge a = 0$ and $\mathrm{coz}(\alpha - \mathbf{1}) \le b$. Such a map can be chosen to satisfy $\mathbf{0} \le \alpha \le \mathbf{1}$ when it exists.

Proof . (1) ⇒ (2). By Theorem 3.6 (5) is clear.

(2) ⇒ (3). Since $\mathrm{Coz}_c L \subseteq \mathrm{Coz}L$, it is obvious.

(3) ⇒ (4). Take $\varphi \in \mathcal{R}L$ and $\delta \in \mathcal{R}_c(L)$ such that $c = \mathrm{coz}(\varphi)$ and $d = \mathrm{coz}(\delta)$. Then Lemma 3.8 (2) shows that $\varphi = \varphi\rho\delta^2$ for some invertible element $\rho \in \mathcal{R}_c(L)$. Putting $\alpha = \mathbf{1} - \rho\delta^2$, then we have $\alpha \in \mathcal{R}_c(L)$ such that

$$\mathrm{coz}(\alpha) \wedge a \le \mathrm{coz}(\alpha) \wedge c = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\varphi) = \mathrm{coz}(\alpha\varphi) = 0$$

and

$$\mathrm{coz}(\alpha - \mathbf{1}) = \mathrm{coz}(-\rho\delta^2) = \mathrm{coz}(-\rho) \wedge \mathrm{coz}(\delta^2) = 1 \wedge \mathrm{coz}(\delta) = \mathrm{coz}(\delta) = d \leq b.$$

(4) $\Rightarrow$ (1). Let α satisfies (4). Define $x_0 = a$, $x_1 = b$, and $x_q = \alpha(-, q)$ for $q \in (0,1) \cap \mathbb{Q}$. We claim that the subset $\{x_q : q \in [0,1] \cap \mathbb{Q}\} \subseteq L$ is a c-scale between a and b. That is because:

a: $x_q = \alpha(-, q) \in \mathrm{Coz}_c L$ for $q \in (0,1) \cap \mathbb{Q}$.

b: $x_p \prec_c x_q$ whenever $p < q$ in $(0,1) \cap \mathbb{Q}$ since $\alpha(p, -) \in \mathrm{Coz}_c L$ with $x_p \wedge \alpha(p,-) = \alpha(-,p) \wedge \alpha(p,-) = 0$ and $x_q \vee \alpha(p,-) = \alpha(-,q) \vee \alpha(p,-) = 1$.

c: $x_0 \prec_c x_q$ whenever $0 < q$ since $\mathrm{coz}(\alpha) \in \mathrm{Coz}_c L$ with $x_0 \wedge \mathrm{coz}(\alpha) = a \wedge \mathrm{coz}(\alpha) = 0$ and $x_q \vee \mathrm{coz}(\alpha) = \alpha(-,q) \vee \alpha\big((-,0) \vee (0,-)\big) = 1$.

d: $x_q \prec_c x_1$ whenever $q < 1$ since $\alpha(q,-) \in \mathrm{Coz}_c L$ with $x_q \wedge \alpha(q,-) = \alpha(-,q) \wedge \alpha(q,-) = 0$ and $x_1 \vee \alpha(q,-) = b \vee \alpha(q,-) \geq \mathrm{coz}(\alpha - \mathbf{1}) \vee \alpha(q,-) = \alpha\big((-,1) \vee (1,-)\big) \vee \alpha(q,-) = 1$. □

An immediate corollary to the foregoing lemma is the following.

Corollary 3.10. Let L be a frame and $a, b \in \mathrm{Coz}_c L$. Then $a \prec\!\!\prec_c b$ if and only if $a \prec_c b$ if and only if $a \prec b$ in $\mathrm{Coz}_c L$.

For the following, recall that if $a \prec\!\!\prec b$ in a frame L, then $b^* \prec\!\!\prec a^*$ in L.

Corollary 3.11. Let L be a frame and $a, b \in L$. If $a \prec\!\!\prec_c b$ in L, then $b^* \prec\!\!\prec_c a^*$ in L.

Proof . Let $a \prec\!\!\prec_c b$ in L. Then Theorem 3.9 shows that there is some $\alpha \in \mathcal{R}_c(L)$ such that $a \wedge \mathrm{coz}(\alpha) = 0$ and $\mathrm{coz}(\alpha - \mathbf{1}) \leq b$. Take $\beta = \mathbf{1} - \alpha$. Then we would have

$$b^* \wedge \mathrm{coz}(\beta) = b^* \wedge \mathrm{coz}(\mathbf{1} - \alpha) = b^* \wedge \mathrm{coz}(\alpha - \mathbf{1}) \leq b^* \wedge b = 0,$$

and

$$\mathrm{coz}(\beta - \mathbf{1}) = \mathrm{coz}(\mathbf{1} - \alpha - \mathbf{1}) = \mathrm{coz}(-\alpha) = \mathrm{coz}(\alpha) \leq a^*.$$

Since $\beta = \mathbf{1} - \alpha \in \mathcal{R}_c(L)$, Theorem 3.9 shows that $b^* \prec\!\!\prec_c a^*$ in L. □

4 *c*-regular frames

A frame L is called *regular* if for every $a \in L$ we have $a = \bigvee_{x \prec a} x$. This motivates the following definition.

Definition 4.1. A frame L is called c-regular if for every $a \in L$ we have

$$a = \bigvee_{x \prec_c a} x.$$

Lemma 4.2. Let L be a compact c-regular frame and $x \prec a \vee b$ in L. Then there exists an element c in L such that $x \prec a \vee c$ and $c \prec_c b$.

Proof . Since $x \prec a \vee b$, we have $x^* \vee (a \vee b) = 1$. Since L is c-regular, $b = \bigvee_{z \prec_c b} z$. Now

$$1 = x^* \vee (a \vee b) = (x^* \vee a) \vee b = (x^* \vee a) \vee \bigvee_{z \prec b} z = \bigvee_{z \prec b} (x^* \vee a) \vee z$$

But L is compact, so there exist $z_1, \ldots, z_n$ in L such that $z_i \prec_c b$ $(1 \leq i \leq n)$ and

$$1 = (x^* \vee a \vee z_1) \vee \cdots \vee (x^* \vee a \vee z_n) = x^* \vee a \vee (z_1 \vee \cdots \vee z_n).$$

Hence $x \prec a \vee (z_1 \vee \cdots \vee z_n)$ and by Theorem 3.2 (6), $z_1 \vee \cdots \vee z_n \prec_c b$. Thus $z_1 \vee \cdots \vee z_n$ is the desired c. □

Corollary 4.3. Let L be a compact c-regular frame. Then, the following statements are true.

(1) If $x \prec b$ in L, then there exists an element c in L such that $x \prec c \prec_c b$.

(2) If $x \prec b$ in L, then there exists an element s in BL such that $x \prec_c s \prec_c b$.
(3) The relations of $\prec$, $\prec\!\!\prec$, $\prec_c$ and $\prec\!\!\prec_c$ are equal.

Proof . (1). Take $a = 0$ and apply the above lemma.

(2). By (1), there exists an element c in L such that $x \prec c \prec_c b$. Hence, there exists an element t in $\mathrm{Coz}_c L$ such that $c \wedge t = 0$ and $t \vee b = 1$. Hence, there exists $\{t_n\}_{n\in\mathbb{N}} \subseteq BL$ such that $t = \bigvee_{n\in\mathbb{N}} t_n$, which implies that there exists an element n in $\mathbb{N}$ such that $c \wedge t_n = c \wedge t = 0$ and $t_n \vee b = t \vee b = 1$. Then $x \prec c \leq t_n^* \prec_c t_n^* \leq b$, which implies that $x \prec_c t_n^* \prec_c b$.

(3). It is obvious by (1) and (2). □

5 c-completely regular frames

Let X be a topological space. Then in the frame $\mathfrak{O}(X)$ we have $U \prec\!\!\prec_c V$ if and only if there exists a continuous map $f : X \to [0, 1]$ with countable image such that $f(U) = 0$, $f(X - V) = 1$. So, if we assume that for every $V \in \mathfrak{O}(X)$ we have $V = \bigvee\limits_{U \prec\!\!\prec_c V} U$ then X will be a c-completely regular space; since for every closed subset F and $x \notin F$, by applying the above assumption for $V = X - F$, we obtain $U \in \mathfrak{O}(X)$ with $x \in U \prec\!\!\prec_c V$, and hence we get a continuous map $f : X \to [0, 1]$ with countable image such that $f(x) = 0$ and $f(F) = 1$.Therefore, a Hausdorff space X is c-completely regular if and only if for any open set V of X, $V = \bigvee\limits_{U \prec\!\!\prec_c V} U$. In addition, recall that a frame L is called *completely regular* if for every $a \in L$ we have $a = \bigvee\limits_{x \prec\!\!\prec a} x$. These motivate the following definition.

Definition 5.1. A frame L is called c-completely regular if for every $a \in L$ we have

$$a = \bigvee_{x \prec\!\!\prec_c a} x.$$

It is clear that a topological space X is c-completely regular if and only if $\mathfrak{O}(X)$ is a c-completely regular frame. Also, any c-completely regular frame is completely regular since $a \prec\!\!\prec_c b$ implies $a \prec\!\!\prec b$.

Recall that a frame L is called *zero-dimensional* if each of its elements is a join complemented elements. In [15], the authors show that the set $\mathrm{Coz}_c L$ is a base for a frame L if and only if L is a zero-dimensional frame. Now, since $a \in BL$ implies $a \prec\!\!\prec_c a$, and $a \prec\!\!\prec_c b$ implies $a \prec\!\!\prec_c s \prec\!\!\prec_c b$ with $s \in \mathrm{Coz}_c L$, the following theorem is immediate.

Theorem 5.2. The following are equivalent for any frame L.

(1) L is a c-completely regular frame.
(2) $\mathrm{Coz}_c L$ is a base for L.
(3) L is a zero-dimensional frame.

Recall from [4] that a *strong inclusion* on a frame L is a binary relation $\triangleleft$ on L such that

(1) If $x \leq a \triangleleft b \leq y$ then $x \triangleleft y$.

(2) $\triangleleft$ is a sublattice of $L \times L$; that is ,$0 \triangleleft 0$, $1 \triangleleft 1$ and if $x \triangleleft a$, $y \triangleleft b$ then $x \vee y \triangleleft a \vee b$, $x \wedge y \triangleleft a \wedge b$.

(3) If $a \triangleleft b$ then $a \prec b$.

(4) If $a \triangleleft b$ then there exists c with $a \triangleleft c \triangleleft b$ (say $\triangleleft$ interpolates).

(5) If $a \triangleleft b$ then $b^* \triangleleft a^*$.

(6) For each $a \in L$, $a = \bigvee\limits_{x \triangleleft a} x$.

Remark 5.3. By Theorem 3.6 and Corollary 3.11, we get that $\prec\!\!\prec_c$ is a strong inclusion on a c-completely regular frame.

Definition 5.4. An ideal I of a frame L is said to be c-completely regular if $a \in I$ implies $a \prec\!\!\prec_c b$ for some $b \in I$.

We denote the set of all c-completely regular ideals of L by $\mathrm{c-CRegId}(L)$. Then $\mathrm{c-CRegId}(L) \subseteq \beta L$, the Stone-Čech compactification of L.

Lemma 5.5. For any frame L, the assignment $a \mapsto \{x \in L : x \prec\!\!\prec_c a\}$ defines a map $r_c : L \to \mathrm{c-CRegId}(L)$ such that

(1) $x \prec\!\!\prec_c a$ if and only if $r_c(x) \prec r_c(a)$ in $\mathrm{c-CRegId}(L)$.
(2) For each a, $r_c(a) = \bigvee_{x \prec\!\!\prec_c a} r_c(x)$.
(3) For each a, $r_c(a) = \bigvee\{I \in \mathrm{c-CRegId}(L) : I \prec r_c(a)\}$.
(4) r_c is a right adjoint to $\bigvee$.
(5) For any $a \in L$, we have $r_c(a^*) = \big(r_c(a)\big)^*$.
(6) For any $a, b \in \mathrm{Coz}_c L$, we have $r_c(a \vee b) = r_c(a) \vee r_c(b)$.

Proof . First, by conditions (1), (2), (4) of $\prec\!\!\prec_c$, we get that for each a, $r_c(a) \in \mathrm{c-CRegId}(L)$. Hence, r_c is a map. Further

(1) Let $x \prec\!\!\prec_c a$ be given. Then, by Theorem 3.6, there are $u, v \in \mathrm{Coz}_c L$ such that $x \prec\!\!\prec_c u \prec\!\!\prec_c v \prec\!\!\prec_c a$. So

$$v \in r_c(a) \,\text{and}\, x \prec\!\!\prec_c a \,\text{and}\, u \prec v \Rightarrow 1 = u^* \vee v \in r_c(x^*) \vee r_c(a)$$

On the other hand, we get that $r_c(x) \cap r_c(x^*) = \{0\}$. Hence, $r_c(x) \prec r_c(a)$.

Conversely, let $r_c(x) \prec r_c(a)$ in $\mathrm{c-CRegId}(L)$. Then there exists a c-completely regular ideal J such that $r_c(x) \wedge J = \{0\}$ and $r_c(a) \vee J = L$. So, $\bigvee (r_c(x) \wedge J) = 0$, that is, $x \wedge \bigvee J = 0$, and $z \vee t = 1$ for some $z \in r_c(a)$, $t \in J$. Thus $x \wedge t = 0$ and so $x = x \wedge 1 = x \wedge (z \vee t) = (x \wedge z) \vee (x \wedge t) = x \wedge z$. Hence $x \leq z$ and $z \prec\!\!\prec_c a$. So, $x \prec\!\!\prec_c a$.

(2) Let $a \in L$. Then for $x \prec\!\!\prec_c a$, by (1), $r_c(x) \prec r_c(a)$. Hence, $r_c(x) \subseteq r_c(a)$. So $\bigvee_{x \prec\!\!\prec_c a} r_c(x) \subseteq r_c(a)$. On the other hand, for each $x \in r_c(a)$ we have $x \prec\!\!\prec_c a$, and so, by the property (4) of $\prec\!\!\prec_c$, there exists y with $x \prec\!\!\prec_c y \prec\!\!\prec_c a$. So, $x \in r_c(y) \subseteq \bigvee_{x \prec\!\!\prec_c a} r_c(x)$. Thus, $r_c(a) \subseteq \bigvee_{x \prec\!\!\prec_c a} r_c(x)$. Hence, $r_c(a) = \bigvee_{x \prec\!\!\prec_c a} r_c(x)$.

(3) Let $a \in L$. By (2), $r_c(a) = \bigvee_{x \prec\!\!\prec_c a} r_c(x)$. But, by (1), if $x \prec\!\!\prec_c a$ then $r_c(x) \prec r_c(a)$. So,

$$r_c(a) = \bigvee_{x \prec\!\!\prec_c a} r_c(x) \subseteq \bigvee_{r_c(x) \prec r_c(a)} r_c(x) \subseteq \bigvee\{I \in \mathrm{c-CRegId}(L) : I \prec r_c(a)\}$$

But, $\bigvee_{I \prec r_c(a)} I \subseteq r_c(a)$ is true since $I \prec r_c(a)$ implies $I \subseteq r_c(a)$. Thus, $r_c(a) = \bigvee\{I \in \mathrm{c-CRegId}(L) : I \prec r_c(a)\}$.

(4) r_c is a right adjoint to $\bigvee$, since for every c-completely regular ideal J and $a \in L$, we have

$$\bigvee J \leq a \Leftrightarrow J \subseteq r_c(a)$$

because if $\bigvee J \leq a$ and $x \in J$ then $x \prec\!\!\prec_c z$ for some $z \in J$ and hence $x \prec\!\!\prec_c \bigvee J$, which implies $x \prec\!\!\prec_c a$, and if $J \subseteq r_c(a)$ then $\bigvee J \leq \bigvee r_c(a) \leq a$.

(5) Since r_c preserves zero and arbitrary meets, we would have

$$r_c(a) \wedge r_c(a^*) = r_c(a \wedge a^*) = r_c(0) = \{0\},$$

showing that $r_c(a^*) \leq \big(r_c(a)\big)^*$. This establishes the inclusion $\subseteq$. Next, since

$$0 = \bigvee\{0\} = \bigvee \Big(r_c(a) \wedge \big(r_c(a)\big)^*\Big) = \bigvee r_c(a) \wedge \bigvee \big(r_c(a)\big)^* = a \wedge \bigvee \big(r_c(a)\big)^*,$$

we have $\bigvee \big(r_c(a)\big)^* \leq a^*$. Thus, by (4), $\big(r_c(a)\big)^* \leq r_c(a^*)$, proving the other inclusion.

(6) First note that $x \prec_c a \vee b$ implies that $x \prec_c u \vee b$ for some $u \in \mathrm{Coz}_c L$ such that $u \prec_c a$. For this, let $t \in \mathrm{Coz}_c L$ such that $x \wedge t = 0$ and $t \vee a \vee b = 1$. Since $\mathrm{Coz}_c L$ is normal, take $u, v \in \mathrm{Coz}_c L$ such that $a \vee v = 1 = t \vee u \vee b$ and $u \wedge v = 0$ to obtain $x \prec_c u \vee b$, $u \in \mathrm{Coz}_c L$, and $u \prec_c a$. It follows now that $x \prec\!\!\prec_c a \vee b$ implies $x \leq u \vee v$ for suitable $u, v \in \mathrm{Coz}_c L$ such that $u \prec_c a$ and $v \prec_c b$, showing that $r_c(a \vee b) \subseteq r_c(a) \vee r_c(b)$ since $\prec_c = \prec\!\!\prec_c$ in $\mathrm{Coz}_c L$. The revers inclusion is immediate, and so $r_c(a \vee b) = r_c(a) \vee r_c(b)$. □

Lemma 5.6. For any frame L, the following statements are true.

(1) $\mathrm{c-CRegId}(L)$ is a compact regular frame.
(2) If L is c-completely regular, then $\bigvee : \mathrm{c-CRegId}(L) \to L$ is a compactification for L.

Proof . (1). First we show that $\mathrm{c-CRegId}(L)$ is a subframe of $\mathrm{Id}(L)$. Since $0 \prec\!\!\prec_c 0$ and $1 \prec\!\!\prec_c 1$, we get that $\{0\}$ and L are c-completely regular. Now, let $I, J \in \mathrm{c-CRegId}(L)$. Then, for $x \in I \cap J$ there exist $y \in I$ and $z \in J$ such that $x \prec\!\!\prec_c y$ and $x \prec\!\!\prec_c z$. Hence $x \prec\!\!\prec_c y \wedge z$ (since $\prec\!\!\prec_c$ is a sublattice of $L \times L$), where $y \wedge z \in I \cap J$. Thus, $I \cap J \in \mathrm{c-CRegId}(L)$. Also, for $x = y \vee z \in I \vee J$ with $y \in I$, $z \in J$ there exist $s \in I$ and $t \in J$ such that $y \prec\!\!\prec_c s$ and $z \prec\!\!\prec_c t$. Thus $x = y \vee z \prec\!\!\prec_c s \vee t$, where $s \vee t \in I \vee J$. Hence $I \vee J \in \mathrm{c-CRegId}(L)$. Finally, if $D \subseteq \mathrm{c-CRegId}(L)$ is directed then $\bigvee D = \bigcup D \in \mathrm{c-CRegId}(L)$. Therefore, $\mathrm{c-CRegId}(L)$ is a subframe of $\mathrm{Id}(L)$.

Now, since $\mathrm{Id}(L)$ is a compact frame, we conclude that $\mathrm{c-CRegId}(L)$ is compact. Also, for every $J \in \mathrm{c-CRegId}(L)$, we clearly have $J = \bigvee_{r_c(a) \subseteq J} r_c(a)$. So, using part (3) of 5.5, for each $J \in \mathrm{c-CRegId}(L)$, we have

$$J = \bigvee_{r_c(a) \subseteq J} r_c(a) = \bigvee_{r_c(a) \subseteq J} \left(\bigvee \{I \in \mathrm{c-CRegId}(L) : I \prec r_c(a)\}\right) = \bigvee \{I \in \mathrm{c-CRegId}(L) : I \prec J\}.$$

Thus $\mathrm{c-CRegId}(L)$ is a regular frame.

(2) Since for each $a \in L$, $a = \bigvee r_c(a)$, we conclude that $\bigvee : \mathrm{c-CRegId}(L) \to L$ is onto, and hence it is a compactification for L. □

We note that for any a and x in a frame L, if $r_c(x) \prec\!\!\prec_c r_c(a)$ in $\mathrm{c-CRegId}(L)$, then part (1) of Lemma 5.5 implies that $x \prec\!\!\prec_c a$. For the converse, we give the next lemma.

Lemma 5.7. For any frame L, if $\mathrm{c-CRegId}(L)$ is a c-completely regular frame, then for any a and x in a frame L, $x \prec\!\!\prec_c a$ implies that $r_c(x) \prec\!\!\prec_c r_c(a)$ in $\mathrm{c-CRegId}(L)$.

Proof . Let $x \prec\!\!\prec_c a$ be given. Then, by part (1) of Lemma 5.5, $r_c(x) \prec r_c(a)$ in $\mathrm{c-CRegId}(L)$. Since $\mathrm{c-CRegId}(L)$ is a compact c-completely regular frame, we conclude from Corollary 4.3 that $r_c(x) \prec\!\!\prec_c r_c(a)$ in $\mathrm{c-CRegId}(L)$ □

Theorem 5.8. For any frame L, $\mathrm{c-CRegId}(L)$ is a c-completely regular frame if and only if $t \prec\!\!\prec_c a$ implies that $r_c(t) \prec\!\!\prec_c r_c(a)$ in $\mathrm{c-CRegId}(L)$ for any a in a frame L and $t \in \mathrm{Coz}_c L$.

Proof . The 'if' part is true by the forgoing lemma. To prove the 'only if' part, let $I \in \mathrm{c-CRegId}(L)$ and $x \in I$. Then there is an element y in I such that $x \prec\!\!\prec_c y$, and hence, by Corollary 3.7, $x \prec\!\!\prec_c t \prec\!\!\prec_c y$ for some $t \in \mathrm{Coz}_c L$. Hence $x \in r_c(t)$ and by the present hypothesis , $r_c(t) \prec\!\!\prec_c r_c(y)$. But $y \in I$ and hence $r_c(y) \subseteq I$. Thus $r_c(t) \prec\!\!\prec_c r_c(y) \subseteq I$ which implies $r_c(t) \prec\!\!\prec_c I$. Therefore, $I = \bigvee_{J \prec\!\!\prec_c I} J$ as required. □

Let L be a c-completely regular frame. We write $\beta_c L$ for the compactification $\mathrm{c-CRegId}(L)$. It is known that the coreflection map $\bigvee : Id(BL) \to L$ is a compactification for a frame L if and only if L is zero-dimensional. We denote this compactification by $\beta_0 L$. Now, by [4, Page 110], we can infer that $\beta_c L \cong \beta_0 L$.

We now move to open and closed quotients. That is, we aim to show that if L is a c-completely regular frame and $a \in L$, then $\downarrow a$ and $\uparrow a$ are c-completely regular frames. We begin with the following lemma.

Lemma 5.9. Let $f : L \to M$ be any frame map. Then

(1) f preserves $\prec_c$.
(2) f preserves $\prec\!\!\prec_c$.
(3) If I is a c-completely regular ideal of L, then $< f(I) >$ is a c-completely regular ideal of M.
(4) If L is c-completely regular, then $f(L)$ is c-completely regular.

Proof . (1). Let $a, b \in L$ with $a \prec_c b$ be given. Then there exists $x \in \mathrm{Coz}_c L$ such that $a \wedge x = 0$ and $x \vee b = 1$. Then there exists a family $\{x_n\}_{n \in \mathbb{N}} \subseteq BL$ such that $x = \bigvee_{n=1}^{\infty} x_n$. Since $\{f(x_n)\}_{n \in \mathbb{N}} \subseteq BL$, $f(x) = \bigvee_{n=1}^{\infty} f(x_n) \in \mathrm{Coz}_c(L)$, $f(a) \wedge f(x) = 0$ and $f(x) \vee f(b) = 1$, we conclude that $f(a) \prec_c f(b)$.

(2). Let $a, b \in L$ with $a \prec\!\!\prec_c b$ be given. Then there exists a c-scale $\{x_q : q \in [0,1] \cap \mathbb{Q}\}$ between a and b. By part (1), $\{f(x_q) : q \in [0,1] \cap \mathbb{Q}\}$ is a c-scale between $f(a)$ and $f(b)$. Hence, $f(a) \prec\!\!\prec_c f(b)$.

(3). Let I be a c-completely regular ideal of L and $x \in < f(I) >$. Then there exists $a \in I$ with $x \leq f(a)$. Since $a \in I$, there exists $z \in I$ with $a \prec\!\!\prec_c z$. Now, using (2), $x \leq f(a) \prec\!\!\prec_c f(z)$. Hence $x \prec\!\!\prec_c f(z)$, where $f(z) \in < f(I) >$, which shows that $< f(I) >$ is a c-completely regular ideal.

(4). By part (2), it is evident. □

The above lemma allow us to obtain the following theorem.

Theorem 5.10. Let L be a c-completely regular frame and $a \in L$.Then $\downarrow a$ and $\uparrow a$ are c-completely regular frames.

We close this section by the following proposition.

Proposition 5.11. If L is compact c-completely regular, then $J = r_c(\bigvee J)$ for every $J \in \mathrm{c-CRegId}(L)$.

Proof . For $x \in J$, there exists $z \in J$ such that $x \prec\!\!\prec_c z \leq \bigvee J$, which implies that $x \in r_c(\bigvee J)$. Hence, $J \subseteq r_c(\bigvee J)$. Let $x \in r_c(\bigvee J)$ be given. Then

$$\begin{aligned} x \prec\!\!\prec_c \bigvee J &\Rightarrow x \prec_c \bigvee J \\ &\Rightarrow 1 = x^* \vee \bigvee J = \bigvee_{z \in J} x^* \vee z \\ &\Rightarrow \exists z \in J(1 = x^* \vee z) \\ &\Rightarrow \exists z \in J(x \prec z) \\ &\Rightarrow \exists z \in J(x \leq z) \\ &\Rightarrow x \in J. \end{aligned}$$

Consequently, $J = r_c(\bigvee J)$. □

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