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The "other" Kolmogorov inequality in Riesz spaces

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Abstract

We establish a maximal probability inequality for a class of random variables in the framework of measure-free, Riesz spaces. In the "other" Kolmogorov's inequality, we consider an upper bound for independent random variables and estimate the lower bound for the sums of random variables in Riesz spaces setting. Furthermore, we get an upper bound for the variance of the sum of random variables.

Keywords: Maximal probability inequality; Independent; Conditional expectation; Weak order unit.
MSC 2020: Primary: 46A40; Secondary: 46B40, 60E15, 60B11.

1 Introduction

One of the most absorbing applications in the theory of Riesz spaces is the study of stochastic processes and probability theory. The probability theory of Riesz spaces has a connection with arguments in the classical setting of probability theory. In fact, order theoritic concepts have replaced measure theoritic concepts. Events are traditionally defined as measurable sets and they are associated with band projections. Hence the σ-algebra of measurable sets are associated with a Boolean algebra of band projections. Furthermore, random variables are traditionally defined in terms of measurable functions, hence it is a natural assumption to consider them as members of a Riesz space. The general theory of stochastic processes in Riesz spaces have been considered by Kuo, Labuschagne and Watson, Troitsky and Gessesse, Grobler and Vardy in [17, 18, 26, 23, 10, 11, 9, 27, 12, 16, 14]. It have also been studied in [3, 15, 7]. In the probability theory of Riesz spaces, the role of the probability measure is played by a conditional expectation operator defined in [17, 18].
The probability inequalities play an important role in Probability theory. Since estimation the probabilities of an event or the sum of random variables, helps us to understand the behavior of the elements. Some of important inequalities, such as Burkholder inequalities, Chebyshev's inequality, Jensen's inequality, Doob's martingale inequalities, etc., have been studied in the probability theory of Riesz spaces setting (see [3, 12, 13, 20, 21]). Recently, we have established Kolmogorov's inequality, Hájek-Rényi inequality, Lévy's inequality, Etemadi's inequality, Cantelli's inequality, Skorohod's or Ottaviani's inequality in the probability theory of Riesz spaces setting (see [6, 8]).

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In this paper, we are going to establish a maximal probability inequality for a class of random variables in the framework of measure-free, Riesz spaces. In the "other" Kolmogorov's inequality, we want to get a lower bound for the sums of certain independent random variables in Riesz spaces. Note that in the Kolmogorov's inequality ([7], Theorem 3.1), we obtained an upper bound on the sums of certain independent random variables.

2 Preliminaries

We imagine that the readers are familiar to the basic notations of Riesz spaces [1, 29], but for the convenience of the readers, we express some of the main concepts.
Let $\mathscr{E}$ be a Dedekind complete Riesz space with a weak order unit E. The Riesz space $\mathscr{E}$ is called *laterally complete* if every subset of $\mathscr{E}$, which consists of mutually disjoint elements, has a supremum in $\mathscr{E}$. Moreover $\mathscr{E}$ is said to be *universally complete* if it is laterally complete and Dedekind complete. A *universal completion* of $\mathscr{E}$ which denoted by $\mathscr{E}^u$ is a universally complete space that contains $\mathscr{E}$ as an order dense ideal. A strictly positive order-continuous projection $\mathbb{F}$ on $\mathscr{E}$ mapping weak order units to weak order units and having Dedekind complete range is called a *conditional expectation operator* on $\mathscr{E}$. The Riesz space $\mathscr{E}$ is called $\mathbb{F}$-*universally complete* if for each increasing net (X_α) in $\mathscr{E}_+$ with $(\mathbb{F}X_\alpha)$ order bounded in $\mathscr{E}^u$, we have that (X_α) is order convergent in $\mathscr{E}$ to X. It is known that $\mathscr{E}_E$, the principal ideal generated by E, is lattice isomorphic to a space $C(K)$ with K as a compact Hausdorff space, such that E corresponds to the constant random variable 1 (see [1]). Note that $\mathscr{E}_E$ can get an f-algebra structure which coincided with the f-algebra structure of $C(K)$. This multiplication can be uniquely extended to $\mathscr{E}^u$, in which E is both a multiplicative unit and a weak order unit. This multiplication is constructed by setting

$$(\mathbb{P}E).(\mathbb{Q}E) = \mathbb{P}\mathbb{Q}E = (\mathbb{Q}E).(\mathbb{P}E),$$

for two band projections $\mathbb{P}$ and $\mathbb{Q}$. For more details about f-algebras one can see [4, 5, 29]. Let X and Y be two elements of $\mathscr{E}$. We define $\mathcal{B}(X \le Y)$ to be the band generated by $(Y-X)^+$ and $\mathcal{B}(Y < X)$ to be its disjoint element. Let $\mathbb{F}$ be a conditional expectation on $\mathscr{E}$ with the range $\mathfrak{F}$. Two band projections $\mathbb{P}$ and $\mathbb{Q}$ in the Boolean algebra $\mathfrak{B}$ of all band projections of $\mathscr{E}$ are called $\mathbb{F}$-*conditionally independent* whenever

$$\mathbb{F}\mathbb{P}\mathbb{F}\mathbb{Q}E = \mathbb{F}\mathbb{P}\mathbb{Q}E = \mathbb{F}\mathbb{Q}\mathbb{F}\mathbb{P}E.$$

Equivalently, $\mathbb{P}$ and $\mathbb{Q}$ are $\mathbb{F}$-conditionally independent whenever

$$\mathbb{F}\mathbb{P}\mathbb{F}\mathbb{Q}|_{\mathfrak{F}} = \mathbb{F}\mathbb{P}\mathbb{Q}|_{\mathfrak{F}} = \mathbb{F}\mathbb{Q}\mathbb{F}\mathbb{P}|_{\mathfrak{F}}.$$

In addition, two elements X and Y in $\mathscr{E}$ are called $\mathbb{F}$-conditionally independent if and only if two order closed Riesz subspaces $< X, \mathcal{R}(\mathbb{F}) >$ and $< Y, \mathcal{R}(\mathbb{F}) >$ generated by X and $\mathcal{R}(\mathbb{F})$ and by Y and $\mathcal{R}(\mathbb{F})$ are $\mathbb{F}$-conditionally independent. By Radon-Nikodým-Douglas-Andô type theorem was established in [28] a subset G of an $\mathbb{F}$-universally Dedekind complete Riesz space $\mathscr{E}$ is a closed Riesz subspace of $\mathscr{E}$ with $\mathcal{R}(\mathbb{F}) \subset G$ if and only if there is a unique conditional expectation $\mathbb{F}_G$ on $\mathscr{E}$ with $\mathcal{R}(\mathbb{F}_G) = G$ and $\mathbb{F}_G\mathbb{F} = \mathbb{F} = \mathbb{F}\mathbb{F}_G$. In a consequence of this theorem, two closed Riesz subspaces $\mathscr{E}_1$ and $\mathscr{E}_2$ with $\mathcal{R}(\mathbb{F}) \subset \mathscr{E}_1 \cap \mathscr{E}_2$ are $\mathbb{F}$-conditionally independent if and only if

$$\mathbb{F}_1\mathbb{F}_2 = \mathbb{F} = \mathbb{F}_2\mathbb{F}_1. \tag{2.1}$$

The equality (2.1) is equivalent to

$$\mathbb{F}_i X = \mathbb{F}X \qquad \text{for all } X \in \mathscr{E}_{3-i}, \quad i = 1, 2. \tag{2.2}$$

For more information about independence see [19, 27, 22]. The domain of a conditional expectation $\mathbb{F}$ can be extended to the *natural domain* $\mathcal{L}^1(\mathbb{F})$, which is an $\mathbb{F}$-universally Dedekind complete ideal of the universal completion $\mathscr{E}^u$ (see [18]). This extension $\mathbb{F}'$, is a conditional expectation on $\mathcal{L}^1(\mathbb{F})$. We shall always identify $\mathbb{F}$ and $\mathbb{F}'$ in this paper without further mention. Note that $\mathcal{L}^1(\mathbb{F})^u = \mathscr{E}^u$. Since $\mathscr{E}^u$ has an f-algebra structure, thus the multiplication of elements of $\mathscr{E}$ is defined but it is not necessarily an element of $\mathscr{E}$. This leads Labuschagne and Watson to define

$$\mathcal{L}^2(\mathbb{F}) := \{X \in \mathcal{L}^1(\mathbb{F}) | X^2 \in \mathcal{L}^1(\mathbb{F})\}.$$

If $X, Y \in \mathcal{L}^2(\mathbb{F})$, then $0 \le (X \pm Y)^2 = X^2 \pm 2XY + Y^2$. Therefore $\pm 2XY \le X^2 + Y^2$ and $2|XY| \le X^2 + Y^2 \in \mathcal{L}^1(\mathbb{F})$. Hence $XY \in \mathcal{L}^1(\mathbb{F})$ (see [23]).
We will use the following results of ([16], Lemma 4.1, Theorem 4.2).

Lemma 2.1. Let $X, Y \in \mathcal{L}^2(\mathbb{F})$ be $\mathbb{F}$-conditionally independent, then

$$\mathbb{F}XY = \mathbb{F}X\mathbb{F}Y = \mathbb{F}Y\mathbb{F}X. \tag{2.3}$$

Theorem 2.2. Let $(X_k)_{k\in\mathbb{N}}$ be an $\mathbb{F}$-conditionally independent sequence in $\mathcal{L}^2(\mathbb{F})$. Then,

$$\mathrm{var}\big(\sum_{k=1}^{n} X_k\big) = \sum_{k=1}^{n} \mathrm{var}(X_k). \tag{2.4}$$

3 Maximal inequalities

In this section, the “other” Kolmogorov's inequality is established in the framework of Riesz spaces. We assume throughout that $\mathbb{F}$ is a conditional expectation on $\mathbb{F}$-universally Dedekind complete Riesz space $\mathcal{L}^1(\mathbb{F})$ with a weak order unit E, such that $\mathbb{F}(E) = E$. For $1 \leq k \leq n$, we set $S_k := \sum_{i=1}^{k} X_i$.
In the Kolmogorov's inequality ([7], Theorem 3.1), we found an upper bound on the probability for the partial sums of a sequence of independent random variables. In fact, if $X_1, X_2, \dots X_n \in \mathcal{L}^2(\mathbb{F})$ are $\mathbb{F}$-independent with $\mathbb{F}X_k = 0$ for every k. Then, for $\lambda > 0$,

$$\lambda^2 \mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n}|S_k| - \lambda E)^+} E \leq \sum_{k=1}^{n} \mathbb{F}{X_k}^2. \tag{3.1}$$

A sketch of the proof : Suppose that $S := \sup_{1\leq k\leq n}|S_k|$. Take

$$\begin{aligned}
&\mathcal{B}_1 := \mathcal{B}(|S_1| \geq \lambda E),\\
&\mathcal{B}_2 := \mathcal{B}(|S_1| < \lambda E, |S_2| \geq \lambda E),\\
&\vdots\\
&\mathcal{B}_n := \mathcal{B}(|S_1| < \lambda E, \dots, |S_{n-1}| < \lambda E, \dots, |S_n| \geq \lambda E),
\end{aligned} \tag{3.2}$$

with corresponding band projections $\mathbb{P}_{\mathcal{B}_k}$, for $1 \leq k \leq n$. If $\mathbb{P}$ denotes the projection onto the band $\mathcal{B}(S \geq \lambda E)$, we have that $\mathbb{P} = \sum_{k=1}^{n} \mathbb{P}_{\mathcal{B}_k}$. In the projection band $\mathcal{B}_k$, $\mathbb{P}_{\mathcal{B}_k} S_k^2 \geq \mathbb{P}_{\mathcal{B}_k} \lambda^2 E = \lambda^2 \mathbb{P}_{\mathcal{B}_k} E$, hence

$$\lambda^2 \mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n}|S_k| - \lambda E)^+} E = \lambda^2 \sum_{k=1}^{n} \mathbb{F}\mathbb{P}_{\mathcal{B}_k} E \leq \sum_{k=1}^{n} \mathbb{F}\mathbb{P}_{\mathcal{B}_k} {S_k}^2 = \mathbb{F}\mathbb{P}{S_k}^2 \leq \mathbb{F}{S_k}^2. \tag{3.3}$$

Since $\mathbb{F}$ is linear we get that

$$\mathbb{F}{S_k}^2 = \mathbb{F}({S_k}^2 - {S_n}^2) + \mathbb{F}{S_n}^2 = \sum_{i=1}^{n} \sum_{j=k+1}^{n} \mathbb{F}X_iX_j - \sum_{i=k+1}^{n} \mathbb{F}X_i^2 + \mathbb{F}{S_n}^2.$$

Thus

$$\begin{aligned}
\sum_{k=1}^{n} \mathbb{F}\mathbb{P}_k {S_k}^2 &\leq \sum_{i=1}^{n} \sum_{j=k+1}^{n} \mathbb{F}X_iX_j - \sum_{i=k+1}^{n} \mathbb{F}X_i^2 + \mathbb{F}{S_n}^2\\
&= \sum_{i=1}^{n} \sum_{j=k+1}^{n} \mathbb{F}X_i\mathbb{F}X_j - \sum_{i=k+1}^{n} \mathbb{F}X_i^2 + \mathbb{F}{S_n}^2 \qquad (X_i,\ X_j \text{ are } \mathbb{F}\text{-independent})\\
&= -\sum_{i=k+1}^{n} \mathbb{F}X_i^2 + \mathbb{F}{S_n}^2 \qquad (\mathbb{F}(X_k) = 0 \text{ for } 1 \leq k \leq n)\\
&\leq \mathbb{F}{S_n}^2 = \sum_{k=1}^{n} \mathbb{F}{X_k}^2.
\end{aligned} \tag{3.4}$$

(3.3) together with (3.4) implies that

$$\lambda^2 \mathbb{F}\mathbb{P}_{(\sup_{1\le k\le n}|S_k|-\lambda E)^+}E \le \sum_{k=1}^{n} \mathbb{F}{X_k}^2.$$

We are going to consider the lower bound of Kolmogorov's inequality in Riesz spaces. In the next theorem, we consider an upper bound for $\sup_{1\le k\le n}|X_k|$ and estimate the lower bound of (3.1) as follows.

Theorem 3.1. (The "other" Kolmogorov's inequality)
Let $X_1, X_2, \dots X_n \in \mathcal{L}^2(\mathbb{F})$ be $\mathbb{F}$-independent with $\mathbb{F}X_k = 0$ for every k, and there exists constant $M > 0$, such that

$$\sup_{1\le k\le n}|X_k| \le M.$$

Then for $\lambda > 0$,

$$\sum_{k=1}^{n} \mathbb{F}{X_k}^2 \ \mathbb{F}\mathbb{P}_{(\sup_{1\le k\le n}|S_k|-\lambda E)^+}E \ge \sum_{k=1}^{n} \mathbb{F}{X_k}^2 - (\lambda+M)^2.$$

Proof . Let $\{\mathcal{B}_k : 1 \le k \le n\}$ be defined by (3.2). Define

$$\begin{aligned}
&\mathcal{A}_1 := \mathcal{B}(|S_1| < \lambda E),\\
&\mathcal{A}_2 := \mathcal{B}(|S_1| < \lambda E, |S_2| < \lambda E),\\
&\vdots\\
&\mathcal{A}_n := \mathcal{B}(|S_1| < \lambda E, \dots, |S_{n-1}| < \lambda E, \dots, |S_n| < \lambda E),
\end{aligned}$$

with corresponding band projections $\mathbb{P}_{\mathcal{A}_k}$. Then for all k, $\mathcal{A}_k \cap \mathcal{B}_k = \emptyset$ and

$$\mathbb{P}_{\mathcal{A}_k^d} = \sum_{j=1}^{k} \mathbb{P}_{\mathcal{B}_j}, \tag{3.5}$$

in which $\mathbb{P}_{\mathcal{A}_k^d}$ and $\mathbb{P}_{\mathcal{B}_j}$ are the corresponding band projections with respect to the bands A_k^d and $\mathcal{B}_j$. By (3.5) we obtain

$$\mathbb{P}_{\mathcal{A}_k} = I - \mathbb{P}_{\mathcal{A}_k^d} = I - \sum_{j=1}^{k} \mathbb{P}_{\mathcal{B}_j},$$

hence

$$\mathbb{P}_{\mathcal{A}_k}S_k + \mathbb{P}_{\mathcal{B}_k}S_k = (I - \sum_{j=1}^{k}\mathbb{P}_{\mathcal{B}_j})(S_k) + \mathbb{P}_{\mathcal{B}_k}S_k = (I - \sum_{j=1}^{k-1}\mathbb{P}_{\mathcal{B}_j})(S_k) = \mathbb{P}_{\mathcal{A}_{k-1}}S_k.$$

Thus

$$\mathbb{P}_{\mathcal{A}_k}S_k + \mathbb{P}_{\mathcal{B}_k}S_k = \mathbb{P}_{\mathcal{A}_{k-1}}S_k = \mathbb{P}_{\mathcal{A}_{k-1}}S_{k-1} + \mathbb{P}_{\mathcal{A}_{k-1}}X_k. \tag{3.6}$$

By squaring and taking expectations the first equality in (3.6) yields

$$\mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_k)^2 = \mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_{k-1} + \mathbb{P}_{\mathcal{A}_{k-1}}X_k)^2. \tag{3.7}$$

We have

$$\begin{aligned}
\mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_{k-1})^2 + \mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}X_k)^2 &= \mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_{k-1})^2 + \mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}X_k)^2\\
&\quad + 2\mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_{k-1})\mathbb{F}(X_k)\\
&= \mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_{k-1})^2 + \mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}X_k)^2\\
&\quad + 2\mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_{k-1}X_k) \qquad \text{(by } \mathbb{F}\text{-independent)}\\
&= \mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_{k-1} + \mathbb{P}_{\mathcal{A}_{k-1}}X_k)^2.
\end{aligned} \tag{3.8}$$

Thus by (3.7) and (3.8) we get that

$$\mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_k)^2 = \mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_{k-1})^2 + \mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}X_k)^2. \tag{3.9}$$

Let $\mathbb{P}_k$ denotes the band projection corresponding to the band generated by X_k^2. Then $\mathbb{P}_k X_k^2 = X_k^2$, and we have

$$\begin{aligned}
\mathbb{F}X_k^2\mathbb{F}\mathbb{P}_{\mathcal{A}_{k-1}}E &= \mathbb{F}\mathbb{P}_k X_k^2\mathbb{F}\mathbb{P}_{\mathcal{A}_{k-1}}E \\
&= \mathbb{F}\mathbb{P}_{\mathcal{A}_{k-1}}E\mathbb{F}\mathbb{P}_k X_k^2 && \text{(by $\mathbb{F}$-independent)} \\
&= \mathbb{F}\mathbb{P}_{\mathcal{A}_{k-1}}E\mathbb{P}_k X_k^2 && \text{(by $\mathbb{F}$-independent)} \\
&= \mathbb{F}\mathbb{P}_{\mathcal{A}_{k-1}}EX_k^2 && \text{(by $\mathbb{P}_k X_k^2 = X_k^2$)} \\
&= \mathbb{F}\mathbb{P}_{\mathcal{A}_{k-1}}X_k^2 \\
&= \mathbb{F}\mathbb{P}_{\mathcal{A}_{k-1}}X_k\mathbb{P}_{\mathcal{A}_{k-1}}X_k && \text{(by f-algebra structure)} \\
&= \mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}X_k)^2.
\end{aligned} \tag{3.10}$$

Also by the last equality in (3.6) we get that

$$\begin{aligned}
\mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_k)^2 &= \mathbb{F}(\mathbb{P}_{\mathcal{A}_k}S_k + \mathbb{P}_{\mathcal{B}_k}S_k)^2 \\
&= \mathbb{F}(\mathbb{P}_{\mathcal{A}_k}S_k)^2 + \mathbb{F}(\mathbb{P}_{\mathcal{B}_k}S_k)^2 + 2\mathbb{F}(\mathbb{P}_{\mathcal{A}_k}S_k\mathbb{P}_{\mathcal{B}_k}S_k) \\
&= \mathbb{F}(\mathbb{P}_{\mathcal{A}_k}S_k)^2 + \mathbb{F}(\mathbb{P}_{\mathcal{B}_k}S_k)^2 && \text{(by $\mathcal{A}_k \cap \mathcal{B}_k = \emptyset$)} \\
&= \mathbb{F}(\mathbb{P}_{\mathcal{A}_k}S_k)^2 + \mathbb{F}(\mathbb{P}_{\mathcal{B}_k}S_{k-1} + \mathbb{P}_{\mathcal{B}_k}X_k)^2.
\end{aligned} \tag{3.11}$$

For the band $\mathcal{B}_k$ we have $S_{k-1} < \lambda$ and $\sup_k |X_k| \leq M$, thus by (3.11) we obtain

$$\begin{aligned}
\mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_k)^2 &\leq \mathbb{F}(\mathbb{P}_{\mathcal{A}_k}S_k)^2 + \mathbb{F}(\mathbb{P}_{\mathcal{B}_k}\lambda + \mathbb{P}_{B_k}M)^2 \\
&= \mathbb{F}(\mathbb{P}_{\mathcal{A}_k}S_k)^2 + \mathbb{F}\big(\mathbb{P}_{\mathcal{B}_k}(\lambda + M)\big)^2 \\
&= \mathbb{F}(\mathbb{P}_{\mathcal{A}_k}S_k)^2 + \mathbb{F}(\lambda + M)^2\mathbb{F}\mathbb{P}_{\mathcal{B}_k}E \\
&\leq \mathbb{F}(\mathbb{P}_{\mathcal{A}_k}S_k)^2 + (\lambda + M)^2\mathbb{F}\mathbb{P}_{\mathcal{B}_k}E.
\end{aligned} \tag{3.12}$$

By (3.9) and (3.12) we get

$$\mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_{k-1})^2 + \mathbb{F}X_k^2\mathbb{F}\mathbb{P}_{\mathcal{A}_{k-1}}E \leq \mathbb{F}(\mathbb{P}_{\mathcal{A}_k}S_k)^2 + (\lambda + M)^2\mathbb{F}\mathbb{P}_{\mathcal{B}_k}E,$$

hence,

$$\mathbb{F}X_k^2\mathbb{F}\mathbb{P}_{\mathcal{A}_{k-1}}E \leq \mathbb{F}(\mathbb{P}_{\mathcal{A}_k}S_k)^2 - \mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_{k-1})^2 + (\lambda + M)^2\mathbb{F}\mathbb{P}_{\mathcal{B}_k}E.$$

Note that $\mathbb{P}_{\mathcal{A}_n}E \leq \mathbb{P}_{\mathcal{A}_{k-1}}E$, since $\mathcal{A}_n \subset \mathcal{A}_k$ for all k, thus

$$\mathbb{F}X_k^2\mathbb{F}\mathbb{P}_{\mathcal{A}_n}E \leq \mathbb{F}(\mathbb{P}_{\mathcal{A}_k}S_k)^2 - \mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_{k-1})^2 + (\lambda + M)^2\mathbb{F}\mathbb{P}_{\mathcal{B}_k}E.$$

After summation it implies that

$$\begin{aligned}
\sum_{k=1}^n \mathbb{F}X_k^2\mathbb{F}\mathbb{P}_{\mathcal{A}_n}E &\leq \sum_{k=1}^n \mathbb{F}(\mathbb{P}_{\mathcal{A}_k}S_k)^2 - \sum_{k=1}^n \mathbb{F}(\mathbb{P}_{\mathcal{A}_{k-1}}S_{k-1})^2 + (\lambda + M)^2\sum_{k=1}^n \mathbb{F}\mathbb{P}_{\mathcal{B}_k}E \\
&= \mathbb{F}(\mathbb{P}_{\mathcal{A}_n}S_n)^2 + (\lambda + M)^2\sum_{k=1}^n \mathbb{F}\mathbb{P}_{\mathcal{B}_k}E \\
&= \mathbb{F}(\mathbb{P}_{\mathcal{A}_n}S_n)^2 + (\lambda + M)^2\mathbb{F}\mathbb{P}_{\mathcal{A}_n^d}E.
\end{aligned}$$

Since $S_n < \lambda$, we get that

$$\begin{aligned}
\sum_{k=1}^n \mathbb{F}X_k^2\mathbb{F}\mathbb{P}_{A_n}E &\leq \mathbb{F}(\mathbb{P}_{A_n}\lambda)^2 + (\lambda + M)^2\mathbb{F}\mathbb{P}_{A_n^d}E \\
&\leq \lambda^2\mathbb{F}\mathbb{P}_{A_n}E + (\lambda + M)^2\mathbb{F}\mathbb{P}_{A_n^d}E.
\end{aligned} \tag{3.13}$$

By the fact that $\mathbb{P}_{A_n} = \mathbb{I} - \mathbb{P}_{A_n^d}$, we obtain

$$\begin{aligned}\sum_{k=1}^{n} \mathbb{F}X_k^2 \mathbb{F}\mathbb{P}_{A_n} E &= \lambda^2 \mathbb{F}(\mathbb{I} - \mathbb{P}_{A_n^d})E + (\lambda+M)^2 \mathbb{F}\mathbb{P}_{A_n^d} E \\ &= \lambda^2 - \lambda^2 \mathbb{F}\mathbb{P}_{A_n^d} E + (\lambda+M)^2 \mathbb{F}\mathbb{P}_{A_n^d} E \\ &\leq (\lambda+M)^2.\end{aligned}$$

It means that

$$\sum_{k=1}^{n} \mathbb{F}{X_k}^2 \ \mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n}|S_k| - \lambda E)^-} E \leq (\lambda+M)^2.$$

By the concept of adjoint we get

$$\sum_{k=1}^{n} \mathbb{F}{X_k}^2 \ \mathbb{F}\big(\mathbb{I} - \mathbb{P}_{(\sup_{1\leq k\leq n}|S_k| - \lambda E)^+}\big) E \leq (\lambda+M)^2.$$

Therefore,

$$\sum_{k=1}^{n} \mathbb{F}{X_k}^2 \ \mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n}|S_k| - \lambda E)^+} E \geq \sum_{k=1}^{n} \mathbb{F}{X_k}^2 - (\lambda+M)^2.$$

□

Remark 3.2. Let $\mathbb{F}X_k \neq 0$ and $\sup_k |X_k| \leq M$, then by $0 \leq \mathbb{F} \leq I$ we get that

$$\sup_k \mathbb{F}X_k \leq \mathbb{F}M \leq M,$$

hence $\sup_k |X_k - \mathbb{F}X_k| \leq 2M$. By Theorem 3.1, we obtain

$$\sum_{k=1}^{n} \mathbb{F}{X_k}^2 \mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n}|S_k - \mathbb{F}S_k| - \lambda)^+} E \geq \sum_{k=1}^{n} \mathbb{F}{X_k}^2 - (\lambda+2M)^2.$$

To see another result, by (3.13) in the proof of Theorem 3.1, we get a separate result as follows. In the following corollary, we obtain an upper bound for the variance of the sum of random variables.

Corollary 3.3. Let $X_1, X_2, \ldots X_n \in \mathcal{L}^2(\mathbb{F})$ be $\mathbb{F}$-independent with $\mathbb{F}X_k = 0$ for every k and there exists constant $M > 0$ such that

$$\sup_{1\leq k\leq n} |X_k| \leq M.$$

By (3.13) for $\lambda > 0$,

$$\begin{aligned}\sum_{k=1}^{n} \mathbb{F}{X_k}^2 \ \mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n}|S_k| - \lambda)^-} E &\leq \lambda^2 \mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n}|S_k| - \lambda)^-} E \\ &\quad + (\lambda+M)^2 \mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n}|S_k| - \lambda)^+} E.\end{aligned}$$

Since $\mathbb{P}_{(\sup_{1\leq k\leq n}|S_k| - \lambda)^-} E = I - \mathbb{P}_{(\sup_{1\leq k\leq n}|S_k| - \lambda)^+} E$, we get that

$$\begin{aligned}\sum_{k=1}^{n} \mathbb{F}{X_k}^2 \ \mathbb{F}(I - \mathbb{P}_{(\sup_{1\leq k\leq n}|S_k| - \lambda)^+}) E &\leq \lambda^2 \mathbb{F}(I - \mathbb{P}_{(\sup_{1\leq k\leq n}|S_k| - \lambda)^+}) E \\ &\quad + (\lambda+M)^2 \mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n}|S_k| - \lambda)^+} E.\end{aligned}$$

Take $\mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n}|S_k| - \lambda)^+} E < \delta$, for some $0 < \delta < E$, then

$$(E - \delta) \sum_{k=1}^{n} \mathbb{F}X_k^2 < \lambda^2 (E - \delta) + (\lambda+M)^2 \delta.$$

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