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Iteration Operator Frames: Duality and Stability

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Abstract

The purpose of the paper is to analyze frames $\{f_k\}_{k\in\mathbb{Z}}$ having the form $\{T^k f_0\}_{k\in\mathbb{Z}}$; so called iteration operator frames for some bounded linear operator T and a fixed vector f_0. We state the duality of such frames with respect to their generators. Moreover, we characterize all duals of iteration operator frames with the same structure. We also show that the duals of two iteration operator frames are close to each other provided that the original frames are sufficiently close to each other and vise versa.

Keywords: Frames; Dual frames; Riesz bases; Iteration operator frames.
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1 Introduction

Iteration operator frames are one of the most basic notions in harmonic analysis, which have great importance in practical topics. The new concept of dynamical sampling is introduced by Aldroubi et al. [2] in which the goal is to write the frame in the form of an operator whose operator is a special class of operators such as compact operators, normal operators, self adjoint operators, unitary operator and so on. The subject of this paper is related to frames in the form of an operator with a bounded operator similar to what Christensen and Hasannasab [6] have done.

In the rest of introduction, we will collect some definitions and standard results from frame theory. A sequence $\{f_k\}_{k\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ is called to be a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \le B\|f\|^2, \qquad (f \in \mathcal{H}).$$

If the right inequality in the above relation holds, then $\{f_k\}_{k\in\mathbb{Z}}$ is called a *Bessel* sequence. A sequence $F = \{f_k\}_{k\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ is called a *Riesz sequence* if there are constants $A, B > 0$ so that for all finite sequence of scalars c_k we have

$$A\sum |c_k|^2 \le \left\|\sum_{k\in\mathbb{Z}} c_k f_k\right\|^2 \le B\sum |c_k|^2.$$

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In addition, if F is complete in $\mathcal{H}$, then it is a *Riesz basis* for $\mathcal{H}$. Furthermore, the class of Riesz bases is precisely the class of frames $\{f_k\}_{k\in\mathbb{Z}}$ for which the equation $\sum_{k\in\mathbb{Z}} c_k f_k = 0$, $\{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, forces that $c_k = 0$ for all $k \in \mathbb{Z}$. Frequently the latter condition is expressed by saying that $\{f_k\}_{k\in\mathbb{Z}}$ is ω-independent. This is a much stronger condition that $\{f_k\}_{k\in\mathbb{Z}}$ being linearly independent, which means that if a finite linear combination of vectors from $\{f_k\}_{k\in\mathbb{Z}}$ is zero, all the coefficients must be zero. A frame which is not a Riesz basis is said to be *redundant* or *overcomplete*.

If $F = \{f_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence, its *synthesis operator* $T_F : \ell^2(\mathbb{Z}) \to \mathcal{H}$ is defined by

$$T_F\{c_k\}_{k\in\mathbb{Z}} = \sum_{k\in\mathbb{Z}} c_k f_k.$$

It is well known that T_F is well-defined and bounded. The *excess* of a frame is the number of elements that can be removed yet leaving a frame. It is well known that the excess equals $dim(ker(T_F))$; see [4]. The adjoint operator, $T_F^* : \mathcal{H} \to \ell^2(\mathbb{Z})$, which is called the *analysis operator*, is given by $T_F^* f = \{\langle f, f_k\rangle\}_{k\in\mathbb{Z}}$. Moreover, $S_F := T_F T_F^*$ the *frame operator* of F, is given by

$$S_F f = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle f_k, \quad (f \in \mathcal{H}).$$

If F is a frame, then the frame operator S_F is invertible and $AI_{\mathcal{H}} \le S_F \le BI_{\mathcal{H}}$. The sequence $\widetilde{F} = \{S_F^{-1} f_k\}_{k\in\mathbb{Z}}$, which is also a frame, is called the *canonical dual frame.* A frame $\{g_k\}_{k\in\mathbb{Z}}$ is called a *dual* of $\{f_k\}_{k\in\mathbb{Z}}$ if

$$f = \sum_{k\in\mathbb{Z}} \langle f, g_k\rangle f_k, \qquad (f \in \mathcal{H}).$$

Also, if $F = \{f_k\}_{k\in\mathbb{Z}}$ is a frame, then every dual frame of F is in the form of $\{S_F^{-1} f_k + v_k\}_{k\in\mathbb{Z}}$, where $\{v_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence such that

$$\sum_{k\in\mathbb{Z}} \langle f, f_k\rangle v_k = 0, \qquad (f \in \mathcal{H}). \tag{1.1}$$

In this paper, we consider frames in a Hilbert space $\mathcal{H}$ arising via iterated action of a linear operator T. We say that a frame $\{f_k\}_{k\in\mathbb{Z}}$ is represented by iterated action of a fixed operator on a single element, i.e., it is possible to write

$$\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}. \tag{1.2}$$

Systems of vectors on this form play an important role in mathematical physics, operator theory and modern applied harmonic analysis [5, 8]. Also this appears in the more recent context of dynamical sampling [1, 2]. The Fourier orthonormal basis, single generator shift invariant systems and Gabor systems have the form of (1.2). Also, any Riesz sequence $\{f_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$ has the form $\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}$ for some operator $T \in B(\mathcal{H})$ with closed range [6].

Here, we give a few results which describe the structure of iteration operator frames. The next theorem shows that any frame which is norm-bounded below is a linear union of iterated operators on certain elements.

Theorem 1.1. [6] Consider a frame $\{f_k\}_{k\in\mathbb{Z}}$ which is norm-bounded below. Then the following hold:

(1) The frame $\{f_k\}_{k\in\mathbb{Z}}$ can be decomposed as a finite union

$$\{f_k\}_{k\in\mathbb{Z}} = \bigcup_{j=1}^{J} \{f_k^{(j)}\}_{k\in I_j},$$

where each of the sequences $\{f_k^{(j)}\}_{k\in I_j}$ is an infinite Riesz sequence.

(2) There is a finite collection of vectors from $\{f_k\}_{k=1}^{\infty}$, called $\varphi_1, \ldots, \varphi_j$, and the corresponding bounded operators $T_j : \mathcal{H} \to \mathcal{H}$ which closed range, such that

$$\{f_k\}_{k\in\mathbb{Z}} = \bigcup_{j=1}^{J} \{T_j^k \varphi_j\}_{k\in\mathbb{Z}}.$$

In the next proposition, a necessary and sufficient condition is obtained to write a frame as the iteration operator form. It also implies that, it is independent of the ordering of the elements in $\{f_k\}_{k\in\mathbb{Z}}$.

Proposition 1.2. [6] Consider a frame $\{f_k\}_{k\in\mathbb{Z}}$ which $span\{f_k\}_{k\in\mathbb{Z}}$ is infinite-dimensional. Then the following are equivalent:

(1) The frame $\{f_k\}_{k\in\mathbb{Z}}$ is linearly independent.

(2) There exists a linear operator $T : span\{f_k\}_{k\in\mathbb{Z}} \to \mathcal{H}$ such that $\{f_k\}_{k\in\mathbb{Z}}$ has the form of $\{T^k f_0\}_{k\in\mathbb{Z}}$.

It is worthwhile to mention that the operator T in the above proposition is unique.

The next proposition shows that if a frame $\{f_k\}_{k\in\mathbb{N}\cup\{0\}}$ with finite excess has a representation $\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{N}\cup\{0\}}$, then the operator T is unbounded.

Proposition 1.3. [6] Assume that the frame $\{f_k\}_{k\in\mathbb{N}\cup\{0\}}$ is linearly independent.

(1) If $\{f_k\}_{k\in\mathbb{Z}}$ has finite excess and T is a linear operator such that $\{f_k\}_{k\in\mathbb{N}\cup\{0\}}$ has the form of $\{T^k f_0\}_{k\in\mathbb{N}\cup\{0\}}$, then T is unbounded.

(2) If $\{f_k\}_{k\in\mathbb{N}\cup\{0\}}$ contains a Riesz basis and has infinite excess such that $\{f_k\}_{k\in\mathbb{N}\cup\{0\}}$ has the form of $\{T^k f_0\}_{k\in\mathbb{N}\cup\{0\}}$, then T is unbounded.

Some constructions of a frame of the form $\{T^k f_0\}_{k\in\mathbb{Z}}$ were obtained in [1] and further were discussed in [2]. Throughout this paper we consider only iteration operator frames when their operators are bounded and invertible such as exponential frames [9] and translation frames [10]. It is worthwhile to mention that every iteration operator frame $\{T^k f_0\}_{k\in\mathbb{Z}}$ can be also represented as a representation frame $\{\pi(k)f_0\}_{k\in\mathbb{Z}}$ where $\pi : \mathbb{Z} \to B(\mathcal{H})$, given by $\pi(k)f = T^k f$, is a (not necessary unitary) representation on $\mathbb{Z}$, see [3, 11] for more details of representation frames. We end this section, by keeping in mind representation frames, with a description of complete iteration operator sequences. More precisely, suppose that $T \in B(\mathcal{H})$ is a self-adjoint operator. Then $\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}$ is a complete iteration operator sequence if and only if the only close subspaces of $\mathcal{H}$ containing f_0 that are invariant under T^k for all $k \in \mathbb{Z}$, are $\{0\}$ and $\mathcal{H}$.

2 Duals of iteration operator frames

In this section, we characterize duals of iteration operator frames with the same structure. Here, we assume that $F = \{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{H}$ where $T \in B(\mathcal{H})$ is invertible. The canonical dual of F is the form of $\widetilde{F} = \{(S_F^{-1} T S_F)^k S_F^{-1} f_0\}_{k\in\mathbb{Z}}$. In addition, due to Lemma 3.1 of [7], we obtain

$$T S_F T^* = S_F. \tag{2.1}$$

This easily follows that

$$\begin{aligned} S_F^{-1} f_k &= S_F^{-1} T^k f_0 \\ &= (T^*)^{-k} S_F^{-1} f_0. \end{aligned}$$

In particular, the canonical dual of an iteration operator frame has the iteration operator form. Moreover, (2.1) implies that $\|T\| \geq 1$.

The result stated in the following gives more characterizations of the canonical dual. The proof is easy and left to the reader.

Proposition 2.1. Let $T \in B(\mathcal{H})$ and $F = \{T^k f_0\}_{k\in\mathbb{Z}}$ and $\{T^k g_0\}_{k\in\mathbb{Z}}$ be iteration operator dual frames. The following are equivalent:

(1) $\{(T^*)^{-k} g_0\}_{k\in\mathbb{Z}}$ is the canonical dual of $\{T^k f_0\}_{k\in\mathbb{Z}}$, i.e., $g_0 = S_F^{-1} f_0$.

(2) $\langle f_0, (T^*)^{-k} g_0\rangle = \langle g_0, T^k f_0\rangle, \quad \forall k \in \mathbb{Z}$.

(3) $\langle S_F^{-1} f_0, T^k f_0 \rangle = \langle f_0, (T^*)^{-k} g_0 \rangle, \quad \forall k \in \mathbb{Z}$.

In the next theorem we describe iteration operator duals of an iteration operator frames.

Theorem 2.2. Let $F = \{T^k f_0\}_{k\in\mathbb{Z}}$ be an iteration operator frame and $G = \{g_k\}_{k\in\mathbb{Z}}$ be a dual frame of F. The following are equivalent:

(1) G is an iteration operator frame and it is as the form of $g_k = (S_F^{-1} T S_F)^k g_0$, for all $k \in \mathbb{Z}$.

(2) $\langle g_{n+k}, f_0 \rangle = \langle (S_F^{-1} T S_F)^k g_n, f_0 \rangle$, for all $n, k \in \mathbb{Z}$.

Proof . Since $\{g_k\}_{k\in\mathbb{Z}}$ is a dual of F, so by Lemma 3.3 of [7], $g_k = (S_F^{-1} T S_F)^k g_0$, for all $k \in \mathbb{Z}$ and hence

$$\begin{aligned}
\langle g_{n+k}, f_0 \rangle &= \langle (S_F^{-1} T S_F)^{n+k} g_0, f_0 \rangle \\
&= \langle (S_F^{-1} T S_F)^k (S_F^{-1} T S_F)^n g_0, f_0 \rangle \\
&= \langle (S_F^{-1} T S_F)^k g_n, f_0 \rangle.
\end{aligned}$$

To show (2) $\Rightarrow$ (1) by using (2.1) for every $n \in \mathbb{Z}$ we have

$$\begin{aligned}
g_n &= \sum_{k\in\mathbb{Z}} \left\langle g_n, T^k f_0 \right\rangle S_F^{-1} T^k f_0 \\
&= \sum_{k\in\mathbb{Z}} \left\langle S_F^{-1} T^{-k} S_F g_n, f_0 \right\rangle (T^*)^{-k} S_F^{-1} f_0 \\
&= \sum_{k\in\mathbb{Z}} \langle g_{n-k}, f_0 \rangle (T^*)^{-k} S_F^{-1} f_0 \\
&= \sum_{k\in\mathbb{Z}} \langle g_k, f_0 \rangle (T^*)^{k-n} S_F^{-1} f_0 \\
&= (T^*)^{-n} \sum_{k\in\mathbb{Z}} \langle g_k, f_0 \rangle (T^*)^k S_F^{-1} f_0 \\
&= (T^*)^{-n} \sum_{k\in\mathbb{Z}} \left\langle (S_F^{-1} T S_F)^k g_0, f_0 \right\rangle S_F^{-1} T^{-k} f_0 \\
&= (T^*)^{-n} \sum_{k\in\mathbb{Z}} \left\langle g_0, T^{-k} f_0 \right\rangle S_F^{-1} T^{-k} f_0 \\
&= = (T^*)^{-n} g_0 = (S_F^{-1} T S_F)^n g_0.
\end{aligned}$$

Therefore, G is an iteration operator frame. □

The following result presents a necessary and sufficient condition under which a dual frame of an iteration operator frame has the same structure.

Proposition 2.3. Let $T \in B(\mathcal{H})$, $\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}$ be an iteration operator frame and $\{g_k\}_{k\in\mathbb{Z}} = \{S_F^{-1} f_k + v_k\}_{k\in\mathbb{Z}}$ be a dual frame of F, where $\{v_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence satisfying (1.1). Then $\{g_k\}_{k\in\mathbb{Z}}$ is also an iteration operator frame if and only if $T S_F v_k = S_F v_{k+1}$ for all $k \in \mathbb{Z}$.

Proof . Assume that $\{g_k\}_{k\in\mathbb{Z}}$ is a dual of iteration operator frame of $\{f_k\}_{k\in\mathbb{Z}}$. Applying Theorem 2.2 follows that $g_{k+1} = S_F^{-1} T S_F g_k$, for all $k \in \mathbb{Z}$. So,

$$\begin{aligned}
S_F v_{k+1} &= S_F \left(-S_F^{-1} f_{k+1} + S_F^{-1} f_{k+1} + v_{k+1} \right) \\
&= S_F \left(-S_F^{-1} f_{k+1} + g_{k+1} \right) \\
&= S_F \left(-S_F^{-1} f_{k+1} + S_F^{-1} T S_F (S_F^{-1} f_k + v_k) \right) \\
&= T S_F v_k.
\end{aligned}$$

For the converse, define $U : \mathcal{H} \to \mathcal{H}$ by $U = S_F^{-1} T S_F$. Then, for every $k \in \mathbb{Z}$ we obtain

$$\begin{aligned}
U g_k &= S_F^{-1} T S_F \left(S_F^{-1} f_k + v_k \right) \\
&= S_F^{-1} f_{k+1} + S_F^{-1} T S_F v_k \\
&= S_F^{-1} f_{k+1} + v_{k+1} = g_{k+1}.
\end{aligned}$$

Hence, $\{g_k\}_{k\in\mathbb{Z}}$ has the form of an iteration operator frame. □

In the following theorem we summarize above results and give more conditions on iteration operator duals.

Theorem 2.4. Let $F = \{T^k f_0\}_{k\in\mathbb{Z}}$ be an iteration operator frame and $G = \{g_k\}_{k\in\mathbb{Z}} = \{S_F^{-1} f_k + v_k\}_{k\in\mathbb{Z}}$ be its dual. Consider

(1) $S_F^{-1} T S_F v_k = v_{k+1}, \quad$ for all $k \in \mathbb{Z}$.

(2) $S_F^{-1} T S_F g_k = g_{k+1}, \quad$ for all $\in \mathbb{Z}$.

(3) $\sum_{k\in\mathbb{Z}} \langle f, f_k\rangle v_{k+1} = 0, \quad$ for all $f \in \mathcal{H}$.

(4) $\sum_{k\in\mathbb{Z}} \langle f, f_k\rangle g_{k+1} = S_F^{-1} T S_F f, \quad$ for all $f \in \mathcal{H}$.

Then (1) ⇔ (2), (3) ⇔ (4), (1) ⇒ (3) and (2) ⇒ (4). Moreover, if G is also an iteration operator frame then all of them are equivalent and we have

$$\{g_k\}_{k\in\mathbb{Z}} = \{S_F^{-1} f_k + v_k\}_{k\in\mathbb{Z}} = \{(S_F^{-1} T S_F)^k g_0\}_{k\in\mathbb{Z}}. \tag{2.2}$$

Proof . It is enough to prove the moreover part. First note that if $G = \{U^k g_0\}_{k\in\mathbb{Z}}$ is an iteration operator dual frame, then easily we get

$$UT^* = I. \tag{2.3}$$

Applying (2.1) we obtain

$$U = (T^*)^{-1} = S_F^{-1} T S_F.$$

□

Corollary 2.5. Let $F = \{T^k f_0\}_{k\in\mathbb{Z}}$ and $G = \{U^k g_0\}_{k\in\mathbb{Z}}$ be two iteration operator frames. Then G is a dual of F if and only if $\{(T^*)^k g_0\}_{k\in\mathbb{Z}}$ is a dual of $\{(U^*)^k f_0\}_{k\in\mathbb{Z}}$.

In the following we state the duality of iteration operator frames with respect to their generators.

Theorem 2.6. Let $F = \{T^k f_0\}_{k\in\mathbb{Z}}$ and $G = \{U^k g_0\}_{k\in\mathbb{Z}}$ be two iteration operator frames. Then G is a dual of F if and only if

$$\sum_{k\in\mathbb{Z}} \langle f, f_k\rangle T^k \left(S_F g_0 - f_0\right) = 0$$

Proof . Let $\{g_k\}_{k\in\mathbb{Z}} = \{S_F^{-1} f_k + v_k\}_{k\in\mathbb{Z}}$ be a dual of F such that $\{v_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence which satisfies (1.1). Using Theorem 2.4 follows that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \langle f, f_k\rangle T^k \left(S_F g_0 - f_0\right) &= \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle T^k S_F v_0 \\ &= S_F \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle v_k = 0.\end{aligned}$$

To show the converse, take $v_k = g_k - S_F^{-1} f_k$. Then $\{v_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence. Moreover,

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \langle f, f_k\rangle v_k &= \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle \left(g_k - S_F^{-1} f_k\right) \\ &= \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle \left(U^k g_0 - S_F^{-1} T^k f_0\right) \\ &= \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle \left(S_F^{-1} T^k S_F g_0 - S_F^{-1} T^k f_0\right) \\ &= S_F^{-1} \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle T^k \left(S_F g_0 - f_0\right) \\ &= 0.\end{aligned}$$

□

Corollary 2.7. Let $T \in B(\mathcal{H})$, $\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}$ be an iteration operator frame and $\{g_k\}_{k\in\mathbb{Z}} = \{S_F^{-1} f_k + v_k\}_{k\in\mathbb{Z}}$ be a dual frame of F, where $\{v_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence satisfying (1.1). Then $\{g_k\}_{k\in\mathbb{Z}}$ is also an iteration operator frame if and only if $v_k = T^k w_0$ for some $0 \neq w_0 \in \mathcal{H}$ and in this case, $v_k \neq 0$, for all $k \in \mathbb{Z}$.

3 Perturbation of iteration operator dual frames

In this section, we investigate the properties of iteration operator frames under small perturbations. Assume that F is a frame, we consider a frame G close to F, a small perturbation of G, i.e. there exists $\lambda_1, \lambda_2 \in [0,1)$ such that

$$\left\|\sum_{k\in\mathbb{Z}} c_k(f_k - g_k)\right\| \leq \lambda_1 \left\|\sum_{k\in\mathbb{Z}} c_k f_k\right\| + \lambda_2 \left\|\sum_{k\in\mathbb{Z}} c_k g_k\right\|, \tag{3.1}$$

for all finite scalar sequences $\{c_k\}_{k\in\mathbb{Z}}$.

In the next theorem for an iteration operator frame and its dual we find infinitely many iteration operator duals which are sufficiently close to the original frame.

Theorem 3.1. Let $F = \{T^k f_0\}_{k\in\mathbb{Z}}$ be an iteration operator frame and G its iteration operator dual. For given $\epsilon > 0$, there are infinitely many iteration operator dual frame $W = \{w_k\}_{k\in\mathbb{Z}}$ such that

$$\sum_{k\in\mathbb{Z}} |\langle f, g_k - w_k\rangle|^2 \leq \epsilon \|f\|^2, \quad (f \in \mathcal{H})\,.$$

Proof . Assume that $g_k = S_F^{-1} f_k + v_k$ such that $\{v_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence which satisfies (1.1) and $TS_F v_k = S_F v_{k+1}, k \in \mathbb{Z}$ by Proposition 2.3. Choose $t > 0$ such that $t\left(\sqrt{B_G} + \sqrt{A_F^{-1}}\right)^2 < 1$ and take

$$w_k = S_F^{-1} f_k + (1 - t\epsilon) v_k, \quad k \in \mathbb{Z}.$$

Clearly, $W = \{w_k\}_{k\in\mathbb{Z}}$ is a dual frame of F and

$$S_F^{-1} T S_F (1 - t\epsilon) v_k = (1 - t\epsilon) v_{k+1}.$$

So, W is an iteration operator frame by Proposition 2.3. Moreover, for every $f \in \mathcal{H}$ we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |\langle f, g_k - w_k\rangle|^2 &= t\epsilon \sum_{k\in\mathbb{Z}} |\langle f, v_k\rangle|^2 \\
&= t\epsilon \sum_{k\in\mathbb{Z}} |\langle f, g_k - S_F^{-1} f_k\rangle|^2 \\
&\leq t\epsilon \left(B_G + A_F^{-1} + 2\sqrt{\frac{B_G}{A_F}}\right) \|f\|^2 \\
&\leq \epsilon \|f\|^2.
\end{aligned}$$

This completes the proof. □

Suppose that F and G are two frames. Then

$$\begin{aligned}
\|S_F - S_G\| &= \|T_F T_F^* - T_F T_G^* + T_F T_G^* - T_G T_G^*\| \\
&\leq \|T_F\| \|T_F - T_G\| + \|T_G\| \|T_F - T_G\|.
\end{aligned} \tag{3.2}$$

Hence, by using (3.2) we have

$$\begin{aligned}
\|T_{\widetilde{F}} - T_{\widetilde{G}}\| &= \|S_F^{-1}T_F - S_F^{-1}T_G + S_F^{-1}T_G - S_G^{-1}T_G\| \\
&\leq \|S_F^{-1}\|\|T_F - T_G\| + \|S_F^{-1}(S_G - S_F)S_G^{-1}\|\|T_G\| \\
&\leq \|S_F^{-1}\|\|T_F - T_G\| + \|S_F^{-1}\|\|S_G^{-1}\|\|T_G\|\|T_F - T_G\|\left(\|T_F\| + \|T_G\|\right) \\
&\leq \left(\frac{1}{A_F} + \frac{\sqrt{B_G}\left(\sqrt{B_F} + \sqrt{B_G}\right)}{A_F A_G}\right)\|T_F - T_G\|.
\end{aligned}$$

Using the fact that the canonical dual of $\widetilde{F}$ is F, we obtain

$$\begin{aligned}
\|T_F - T_G\| &\leq \left(\frac{1}{A_{\widetilde{F}}} + \frac{\sqrt{B_{\widetilde{G}}}\left(\sqrt{B_{\widetilde{F}}} + \sqrt{B_{\widetilde{G}}}\right)}{A_{\widetilde{F}} A_{\widetilde{G}}}\right)\|T_{\widetilde{F}} - T_{\widetilde{G}}\| \\
&= \left(B_F + \frac{B_F B_G}{\sqrt{A_G}}\left(\frac{1}{\sqrt{A_F}} + \frac{1}{\sqrt{A_G}}\right)\right)\|T_{\widetilde{F}} - T_{\widetilde{G}}\|
\end{aligned}$$

Hence, the synthesis operator of two iteration operator frames are close to each other if and only if the synthesis operator of their canonical duals are sufficiently close to each other. Moreover, assume that F is a linear independent frame and G is a frame close to F in the sense of (3.1), then G is obviously linearly independent. Combining this fact with Proposition 1.2 shows that every frame sufficiently close to an iteration operator frame has iteration operator structure, see also Proposition 4.1 of [7]. We can summarize these results in the following theorem.

Theorem 3.2. Let F be an iteration operator frame and G be a sequence which satisfies (3.1). Then G is also an iteration operator frame and (3.1) also holds for $\widetilde{F}$ and $\widetilde{G}$.

In the sequal we show that (3.1) holds for a large class of iteration operator dual frames provided that own frames satisfies (3.1)

Corollary 3.3. Let $F = \{T^k f_0\}_{k\in\mathbb{Z}}$ be an iteration operator frame and $\{g_k\}_{k\in\mathbb{Z}}$ a sequence in $\mathcal{H}$ such that (3.1) holds. Then, for every $\epsilon > 0$ and iteation operator dual frames F_1 and G_1 of F and G, respectively, there are infinitely many iteration operator dual frames F^d and G^d of F and G, respectively such that

$$\|T_{F^d} - T_{G^d}\| \leq \lambda_1\|T_{F^d}\| + \lambda_2\|T_{G^d}\| + \epsilon, \tag{3.3}$$

for some $\lambda_1, \lambda_2 \in [0, 1)$.

Proof . The sequence $\{g_k\}_{k\in\mathbb{Z}}$ is also an iteration operator frame by Proposition 4.1 of [7]. Moreover, by using of Theorem 3.2, $\{\widetilde{f}_k\}_{k\in\mathbb{Z}}$ and $\{\widetilde{g}_k\}_{k\in\mathbb{Z}}$ are iteration operator dual frames and there exist $\lambda_1, \lambda_2 \in [0, 1)$ such that

$$\left\|\sum_{k\in\mathbb{Z}} c_k\left(\widetilde{f}_k - \widetilde{g}_k\right)\right\| \leq \lambda_1\left\|\sum_{k\in\mathbb{Z}} c_k\widetilde{f}_k\right\| + \lambda_2\left\|\sum_{k\in\mathbb{Z}} c_k\widetilde{g}_k\right\|. \tag{3.4}$$

Assume that $F_1 = \{S_F^{-1}f_k + u_k\}_{k\in\mathbb{Z}}$ and $G_1 = \{S_G^{-1}g_k + v_k\}_{k\in\mathbb{Z}}$ are iteration operator duals of F and G, respectively where $U = \{u_k\}_{k\in\mathbb{Z}}$ and $V = \{v_k\}_{k\in\mathbb{Z}}$ are Bessel sequences which satisfying in (1.1).

For each $t > 0$ take $F_t^d = \{S_F^{-1}f_k + tu_k\}_{k\in\mathbb{Z}}$ and $G_t^d = \{S_G^{-1}g_k + tv_k\}_{k\in\mathbb{Z}}$. Using Theorem 2.4 follows that F^d and G^d are iteration operator dual frames of F and G, respectively. Applying (3.4) implies that

$$\begin{aligned}
\|T_{F_t^d} - T_{G_t^d}\| &= \|\left(S_F^{-1}T_F + tT_U\right) - \left(S_G^{-1}T_G + tT_V\right)\| \\
&\leq \|T_{\widetilde{F}} - T_{\widetilde{G}}\| + t\|T_U - T_V\| \\
&\leq \lambda_1\|T_{\widetilde{F}}\| + \lambda_2\|T_{\widetilde{G}}\| + t\left(\|T_U\| + \|T_V\|\right) \\
&\leq \lambda_1\|T_{\widetilde{F}} + tT_U\| + \lambda_2\|T_{\widetilde{G}} + tT_V\| \\
&+ \lambda_1 t\|T_U\| + \lambda_2 t\|T_V\| + t\left(\|T_U\| + \|T_V\|\right) \\
&= \lambda_1\|T_{F_t^d}\| + \lambda_2\|T_{G_t^d}\| + t\left[(\lambda_1 + 1)\|T_U\| + (\lambda_2 + 1)\|T_V\|\right]
\end{aligned}$$

Thus, by choosing t small enough, we obtain infinitely many dual frames F_t^d and G_t^d which satisfy (3.3). This completes the proof. □

In the next result, we show that if two iteration operator frames F and G are sufficiently close to each other, then UF and VG are also close to each other in the sense of (3.1) upon some conditions on bounded operators U and V.

Corollary 3.4. Let F and G be two iteration operator frames and $U, V \in B(\mathcal{H})$ be invertible operators such that

$$\|U - V\| + \lambda_2 \|U\| < \|V^{-1}\|^{-1}, \quad \lambda_1 \|U\| \|U^{-1}\| < 1, \tag{3.5}$$

Then UF and VG are also iteration operator frames satisfy (3.1)

Proof . By using (3.1) there exist $\lambda_1, \lambda_1 \in [0, 1)$ we have

$$\begin{aligned}
\|T_{UF} - T_{VG}\| &= \|UT_F - VT_G\| \\
&= \|UT_F + UT_G - UT_G - VT_G\| \\
&= \|U(T_F + T_G) + (U - V)T_\Psi\| \\
&\leq \|U\| (\lambda_1 \|T_F\| + \lambda_2 \|T_G\|) + \|U - V\| \|T_G\| \\
&\leq \|U\| \left(\lambda_1 \|U^{-1}\| \|T_{UF}\| + \lambda_2 \|V^{-1}\| \|T_{VF}\|\right) + \|U - V\| \|V^{-1}\| \|T_{VG}\| \\
&= \lambda_1 \|U\| \|U^{-1}\| \|T_{UF}\| + \|V^{-1}\| (\lambda_2 + \|U - V\|) \|T_{VG}\|
\end{aligned}$$

By using (3.5) both coefficients of $\|T_{UF}\|$ and $\|T_{VG}\|$ belong to $[0, 1)$ and this completes the proof. □

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