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Strongly 2-absorbing subacts over monoids with unique zero

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Abstract

In this article, we introduce (strongly) 2-absorbing ideals of monoids and generalize them to (strongly) 2-absorbing subacts of an act over monoids. Among some useful lemmas, we show that the radical ideal of a strongly 2-absorbing ideal is either a prime ideal or an intersection of two ideals which are only distinct prime ideals minimal over it. Also, we prove that for each strongly 2-absorbing ideal I of a monoid, there exists a strongly 2-absorbing S-act A such that $Ann(A) = I$ and vice versa. Moreover, some of their basic properties are developed.

Keywords: 2-absorbing ideal, 2-absorbing subact, Strongly 2-absorbing ideal, Strongly 2-absorbing subact.
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1 Introduction and Preliminaries

Prime ideals are useful tools in semigroup theory [4]. Recall that, a proper ideal I of S is called *prime* if the inclusion $sSs' \subseteq I$ for $s, s' \in S$, implies that either $s \in I$ or $s' \in I$. A proper ideal I of S is called *semiprime* if the inclusion $sSs \subseteq I$ for $s \in S$ implies that $s \in I$. In equivalent, a proper ideal I is semiprime if and only if the set inclusion $A^2 \subseteq I$ for each ideal A of S implies $A \subseteq I$. A monoid S with zero 0, is said to be *prime* (resp. *semiprime*) if $\{0\}$ is a prime (resp. semiprime) ideal of S. For an ideal I of a monoid S, $\sqrt{I}$ is defined as the intersection of all prime ideals of S containing I. In a commutative monoid, we have $\sqrt{I} = \{t \in S : \exists n \in N,\ t^n \in I\}$; see [7, Proposition 3.2].

Badawi in [2] generalized 2-absorbing ideals from prime ideals in commutative rings. He called a nonzero proper ideal I of a ring R is *2-absorbing* if whenever $r, r', r'' \in R$ and $rr'r'' \in I$, then $rr' \in I$, $rr'' \in I$, or $r'r'' \in I$. He proved that a nonzero proper ideal I of a ring R is 2-absorbing if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then $I_1I_2 \subseteq I$, $I_1I_3 \subseteq I$, or $I_2I_3 \subseteq I$. In other words, he defined 2-absorbing ideals by elements and ideals of rings. On the other hand, according to the additive structure of commutative rings, these concepts defined by elements and ideals are equivalent for rings. Therefore in this research, we intend to define a 2-absorbing ideal of monoids in term of elements, which is a generalization of a prime ideal. But the notion in term of ideals is not equivalent to 2-absorbing ideals. The structural differences of rings $(R, +, .)$ and monoids $(S, .)$ make possible to describe similarities

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and differences of ways to specify a radical ideal of a strongly 2-absorbing ideal of a monoid. Among some following results, we intend to show that if I is a strongly 2-absorbing ideal of a commutative monoid without zero, then either $\sqrt{I}$ is a prime ideal of S or $\sqrt{I} = P_1 \cap P_2$, where P_1 and P_2 are only distinct prime ideals of S, which are minimal over I.

One of the very useful notion in mathematics as well as in computer science is the notion of S-act. Throughout of this paper, S denotes a monoid and A is a right unitary S-act. Recall that a right S-act A is a set A with a function $A \times S \to A$ such that if as is the image of (a, s) for $a \in A$ and $s \in S$, then (i) $(as)t = a(st)$ for $a \in A$ and $s, t \in S$; and (ii) $a1 = a$ for all $a \in A$. An S-subact B of an S-act A written as $B \leq A$, is a subset B of A such that $bs \in B$ for all $b \in B$ and $s \in S$. Thus subacts of the S-act S are ideals of S. A *fixed element* of A, is an element d in an S-act A with $ds = d$ for all $s \in S$. An S-act A is called *centered* if S is a monoid with a two-sided zero element 0 and $|D| = 1$, where D denotes the set of all fixed elements of A. Thus A is a centered if and only if there is a fixed element (necessarily unique) denoted by θ such that: (i) $\theta s = \theta$ for all $s \in S$; and (ii) $a0 = \theta$ for all $a \in A$; the zero of A; see [1]. In what follows, all of the acts are centered unless mentioned otherwise. An equivalence relation ϑ on an S-act A is called a congruence on A if $a_1 \vartheta a_2$ implies $(a_1 s)\vartheta(a_2 s)$ for any $a_1, a_2 \in A$ and $s \in S$. We denote the set of all congruence of A by $Con(A)$. Let B be a subact of an S-act A with a unique fixed element θ. The set $(B : A) = \{s \in S : As \subseteq B\}$ is an ideal of S, which is called *associated ideal* of B and the ideal $Ann(A) = (\Theta : A)$ is called *annihilator* of A, where $\Theta = \{\theta\}$ is a subact of A. A centered S-act A is called *faithful* if $(\Theta : A) = \{0\}$. A subact B of an S-act A is called a *prime* subact if the inclusion $aSs \subseteq B$ for any $a \in A$ and $s \in S$, implies either $a \in B$ or $s \in (B : A)$. In equivalent, a subact B of A is a prime subact if and only if the inclusion $CI \subseteq B$ for any subact C of A and any right ideal I of S, implies that either $C \subseteq B$ or $I \subseteq (B : A)$. Furthermore, a subact B of an act A is called *semiprime* if the inclusion $asSs \subseteq B$ for any $a \in A$ and $s \in S$, implies $as \in B$. An S-act A is called *prime* (resp. *semiprime*) if the subact $\Theta = \{\theta\}$ of A is prime (resp. semiprime) as a subact of A; see [1].

Prime ideals are extended to arbitrary S-acts, analogous to the notion of a prime module which was discussed by Dauns in [6]. We generalize the notion of prime subact to a 2-absorbing subact and investigate some results between a strongly 2-absorbing subact and an associated ideal of it. We will investigate relations between a prime, semiprime, 2-absorbing, and stronly 2-absorbing subact of an act. We show that a monoid S is strongly 2-absorbing if and only if there exists a strongly 2-absorbing faithful S-act, and immediately we conclude Proposition 3.13. We also prove that for an ideal I of a monoid S, I is a strongly 2-absorbing ideal of S if and only if there exists a strongly 2-absorbing S-act A with $(\Theta : A) = I$. It is interesting to find out some results on 2-absorbing subacts which are holding for 2-absorbing submodules introduced as a generalization of prime submodules in [5].

All the notions in this text are standard; we refer the reader to see [3, 4, 8] for more details.

2 Generalizations of prime ideals in monoids

In this section, we define 2-absorbing and strongly 2-absorbing ideals over monoids. We show that there exist at most two prime ideals minimal over a strongly 2-absorbing ideal. We want to show what is the radical ideal of a strongly 2-absorbing ideal.

Definition 2.1. A proper ideal I of a monoid S is called *2-absorbing* whenever the inclusion $sSs'Ss'' \subseteq I$ for any $s, s', s'' \in S$, implies that either $sSs' \subseteq I$ or $s'Ss'' \subseteq I$ or $sSs'' \subseteq I$. A monoid S with zero is called *2-absorbing* if $\{0\}$ is a 2-absorbing ideal of S.

Let S be a commutative monoid. Then the 2-absorbing ideal I is defined as follows: If $ss's'' \in I$ for every $s, s', s'' \in S$, then either $ss' \in I$ or $s's'' \in I$ or $ss'' \in I$.

Definition 2.2. An ideal I of a monoid S is called *strongly 2-absorbing* if the inclusion $ABC \subseteq I$ for any ideals A, B, and C of S implies that either $AB \subseteq I$ or $AC \subseteq I$ or $BC \subseteq I$.

Remark 2.3. Every strongly 2-absorbing ideal of a monoid is a 2-absorbing ideal of it, but the converse is not true (see Example 3.6). Let I be a strongly 2-absorbing ideal of S and let $sSs'Ss'' \subseteq I$ for $s, s', s'' \in S$. Therefore $sSs'Ss''S \subseteq IS \subseteq I$. Since I is a strongly 2-absorbing ideal, we deduce either $sSs'S \subseteq I$ or $sSs''S \subseteq I$ or $s'Ss''S \subseteq I$. As $1 \in S$, then either $sSs' \subseteq I$ or $sSs'' \subseteq I$ or $s'Ss'' \subseteq I$.

We know that an ideal I of S is prime if and only if the set inclusion $AB \subseteq I$ for any ideals A and B of S, implies $A \subseteq I$ or $B \subseteq I$; see [4]. Prime ideals of monoids and 2-absorbing ideals of commutative rings can be defined in two

equivalent ways, by elements and by ideals. But these definitions in term of ideals and in term of elements are not equivalent for 2-absorbing ideals of monoids Thus, this fact is a key for us to define the notion of strongly 2-absorbing ideal. Since every 2-absorbing ideal is not strongly 2-absorbing, we have chosen, strongly 2-absorbing to nominate this new notion.
Example 3.6 shows relationships between prime, semiprime, 2-absorbing, and strongly 2-absorbing ideals of the monoid $(\mathbb{N} \cup \{0\}, .)$. Notice that every ideal of a monoid S is a subact of S_S and vice versa. Therefore this example applies here, too.

Theorem 2.4. If I_1 (resp. I_2) is a 2-absorbing ideal of a monoid S_1 (resp. S_2), then $I_1 \times S_2$ (resp. $S_1 \times I_2$) is a 2-absorbing ideal of $S_1 \times S_2$.

Furthermore, if P and Q are prime ideals of S_1 and S_2, respectively, then $I = P \times Q$ is a 2-absorbing ideal of $S_1 \times S_2$.

Proof . Let $(s,t)(S_1 \times S_2)(s',t')(S_1 \times S_2)(s'',t'') \subseteq I_1 \times S_2$, for $(s,t), (s',t'), (s'',t'') \in S_1 \times S_2$. It implies $sS_1s'S_1s'' \subseteq I_1$. Thus either $sS_1s' \subseteq I_1$ or $sS_1s'' \subseteq I_1$ or $s'S_1s'' \subseteq I_1$, and consequently, either $(s,t)(S_1 \times S_2)(s',t') \subseteq I_1 \times S_2$ or $(s,t)(S_1 \times S_2)(s'',t'') \subseteq I_1 \times S_2$ or $(s',t')(S_1 \times S_2)(s'',t'') \subseteq I_1 \times S_2$.
For the second part, if we have the inclusion $(s_1,t_1)(S_1 \times S_2)(s_2,t_2)(S_1 \times S_2)(s_3,t_3) \subseteq P \times Q$ for $s_1, s_2, s_3 \in S_1$ and $t_1, t_2, t_3 \in S_2$, then $s_1S_1s_2S_1s_3 \subseteq P$ and $t_1S_2t_2S_2t_3 \subseteq Q$. It is obvious that $s_1S_1s_2S_1s_3S_1 \subseteq P$ and $t_1S_2t_2S_2t_3S_2 \subseteq Q$. Since P and Q are prime ideals, then at least one of s_iS_1 and t_iS_2 (for $i = 1, 2, 3$) is a subset of P and Q, respectively. Say $s_1S_1 \subseteq P$ and $t_3S_2 \subseteq Q$. Hence $(s_1,t_1)(S_1 \times S_2)(s_3,t_3)(S_1 \times S_2) \subseteq P \times Q$. Since S_1 and S_2 are monoids, we have $(s_1,t_1)(S_1 \times S_2)(s_3,t_3) \subseteq P \times Q$. □

The next theorem shows that every 1-absorbing ideal is 2-absorbing.

Theorem 2.5. Every prime ideal of a monoid S is a 2-absorbing ideal.

Proof . Let P be a prime ideal of S and let $sSs'Ss'' \subseteq P$, for $s, s', s'' \in S$. Then we have $sSs'Ss''S \subseteq P$. Since P is a prime ideal, then either $sSs'S \subseteq P$ or $s''S \subseteq P$. Therefore we have $sSs' \subseteq P$ or $s'' \in P$, since $1 \in S$. Thus either $sSs' \subseteq P$ or $sSs'' \subseteq P$ or $s'Ss'' \subseteq P$. □

We note that Example 3.6 implies that the converse of above theorem is not true.

Prime and 2-absorbing ideals of a monoid can be generalized to n-absorbing ideal for any positive integer $n \in \mathbb{N}$. An ideal I of S is called n-absorbing whenever for the inclusion $s_1Ss_2S \cdots Ss_{n+1} \subseteq I$, for any $s_1, s_2, \ldots, s_{n+1} \in S$, there are n of the s_i 's whose product $s_1S \cdots Ss_i$ belongs to I. Theorem 2.5 can be achieved for n-absorbing ideals $(n \geq 2)$, by induction on n. It means that every n-absorbing ideal of a monoid is an m-absorbing ideal for any $n \leq m$. The following two lemmas are essential for proving our main results.

Lemma 2.6. Let S' be a proper submonoid of a commutative monoid without zero S, and let P be an ideal of S which is maximal with respect to exclusion of S'. Then P is a prime ideal of S.

Proof . On the contrary, suppose that P is not a prime ideal, then there exist $a, b \in S \setminus P$ such that $ab \in P$. Clearly ideals $\langle P, a\rangle = \bigcup_{i,j \in \mathbb{N}} \{P^i, a^j, P^i a^j\}$ and $\langle P, b\rangle$ contain P strictly and they are generated by P and a and b, respectively. Since P is an ideal of S and is maximal with respect to exclusion of S', then they are not disjoint from S'. Therefore there exist $s, s' \in S'$ such that $s \in \langle P, a\rangle$ and $s' \in \langle P, b\rangle$. If we ponder on forms of their members, then we conclude that there exist $i, j \in \mathbb{N}$ such that $s = a^i$ and $s' = b^j$. Thus $ss' = a^ib^j \in P$, since S is a commutative monoid. Then $ss' \in P \cap S'$, which is a contradiction to maximality of P with respect to exclusion of S'. Hence the assertion follows. □

Lemma 2.7. Let I and P be two ideals of a commutative monoid without zero S and let P be prime. The following statements are equivalent:

(1) P is a minimal prime ideal over I, that is, there exists no prime ideal between I and P;

(2) $S \setminus P$ is a submonoid of S and is maximal with respect to missing I;

(3) For every $x \in P$, there exist $y \in S \setminus P$ and a nonnegative integer n such that $yx^n \in I$.

Proof . (1) ⇒ (2). It is clear that $S \setminus P$ is a submonoid of S which is not contained in I. Consider

$$\Sigma = \{M \mid M \text{ is a submonoid of } S,\ S \setminus P \subseteq M,\ \ M \cap I = \emptyset\}.$$

It is a nonempty set, since $S \setminus P \in \Sigma$. For any chain

$$\cdots \subseteq M_i \subseteq M_{i+1} \subseteq \cdots$$

in Σ, the set $\bigcup_{i \in I} M_i$ is an upper bound. By Zorn's lemma, Σ has a maximal element such as L. Therefore L is a submonoid and maximal respect to missing I and $S \setminus P \subseteq L$. If Q is an ideal containing I that is maximal with respect to being disjoint from L, then by Lemma 2.6, Q is prime. Since Q is disjoint from L, then $L \subseteq S \setminus Q$. As $S \setminus P \subseteq L$, clearly $S \setminus P \subseteq S \setminus Q$ which implies $I \subset Q \subseteq P$, but P is a prime ideal minimal over I which implies that $Q = P$. Since $S \setminus P \subseteq L \subseteq S \setminus Q$, we have $S \setminus P = S \setminus Q = L$.
(2) ⇒ (3). Let $x \in P$, and set

$$T = \{yx^i \mid y \in S \setminus P;\ i = 0, 1, 2, \dots\}.$$

We show that T is a submonoid of S and properly contains $S \setminus P$. Let $yx^i, yx^j \in T$ for some nonnegative integers i and j and $y \in S \setminus P$. As $y^2 \in S \setminus P$, then $y^2x^{i+j} \in T$ and T is a submonoid of S. Since $S \setminus P$ is maximal respect to missing I and $S \setminus P \subset T$, then $T \cap I \neq \emptyset$. Hence there exist an element $y \in S \setminus P$ and a nonnegative integer i such that $yx^i \in I$.
(3) ⇒ (1). Assume that $I \subset Q \subseteq P$, where Q is a prime ideal. If there exists an element $x \in P \setminus Q$, then there exist $y \in S \setminus P \subseteq S \setminus Q$ and a nonnegative integer i such that $yx^i \in I \subset Q$. Now since $x \notin Q$, then $x^i \notin Q$ and also we have $y \notin Q \subseteq P$, which is a contradiction. Therefore $P = Q$ and P is a minimal prime ideal over I. □

In this situation, it seems desirable to have the following theorem.

Theorem 2.8. Let I and P be two ideals of a commutative monoid without zero S. The following statements are equivalent:

(1) P is a minimal prime ideal over I;

(2) For every finitely generated ideal $X \subseteq P$, there exist an ideal $Y \subseteq S \setminus P$ and a nonnegative number n such that $YX^n \subseteq I$.

Proof . (1) ⇒ (2). Let P be a minimal prime ideal over I and let X be a finitely generated ideal of S such that $X = \ < x_1, x_2, \dots, x_r > \subseteq P$. Then by Lemma 2.7, there exist $y_1, y_2, \dots, y_r \in S \setminus P$ and nonnegative integers $n_1, n_2, \dots, n_r$ such that $y_1x_1^{n_1} \in I,\ \ y_2x_2^{n_2} \in I,\ \dots,\ \ y_rx_r^{n_r} \in I$. If $n = \sum_{i=1}^{r} n_i$ and $Y = y_1y_2 \cdots y_rS$, then we can conclude $YX^n \subseteq I$.
(2) ⇒ (1). It follows from Lemma 2.7. □

Now we are going to discuss about the number of prime ideals minimal over a 2-absorbing ideal.

Theorem 2.9. If I is a strongly 2-absorbing ideal of a commutative monoid without zero S, then there exist at most two prime ideals minimal over I.

Proof . On the contrary, suppose that P_1, P_2, and P_3 are three prime ideals of S being minimal over I. Then by Theorem 2.8, there exist finitely generated ideals X_1 and X_2 of S, such that $X_1 \subseteq P_1$, $X_2 \subseteq P_2$, $X_1 \nsubseteq P_2$, $X_1 \nsubseteq P_3$, $X_2 \nsubseteq P_1$ and $X_2 \nsubseteq P_3$. By Theorem 2.8, there exist ideals $Y_2 \nsubseteq P_1$ and $Y_1 \nsubseteq P_2$ such that $Y_2X_1^n \subseteq I$ and $Y_1X_2^m \subseteq I$, for some n, $m \geq 1$ where n and m are the least integers with this property. Since $X_1 \nsubseteq P_2$, $X_2 \nsubseteq P_1$, and I is a strongly 2-absorbing ideal of S and m and n are the least integers, we conclude $Y_2X_1 \subseteq I$ and $Y_1X_2 \subseteq I$. Thus $(Y_1 \cup Y_2)X_1X_2 \subseteq I$. Facts $Y_1 \cup Y_2 \nsubseteq P_1$ and $Y_1 \cup Y_2 \nsubseteq P_2$, imply $(Y_1 \cup Y_2)X_1 \nsubseteq I$ and $(Y_1 \cup Y_2)X_2 \nsubseteq I$. Therefore, $X_1X_2 \subseteq I \subseteq P_3$, which is a contradiction to choosing X_1 and X_2 which are not contained in the prime ideal P_3. Hence there exist at most two minimal prime ideals over I.□

Now by foregone following conclusions, we can determine the radical ideal of a strongly 2-absorbing ideal I in a commutative monoid S as the intersection of all prime ideals of S containing I.
We recall the following lemma for commutative monoids, which can be proved analogously with the same proof of Theorem 2.1 in [2].

Lemma 2.10. Let S be a commutative monoid and let I be a 2-absorbing ideal of S. Then $\sqrt{I}$ is a 2-absorbing ideal and $x^2 \in I$ for every $x \in \sqrt{I}$.

Theorem 2.11. If I is a strongly 2-absorbing ideal of a commutative monoid S without any zero, then one of the following statements holds:

(1) $\sqrt{I} = P$ is a prime ideal of S such that $P^2 \subseteq I$.

(2) $\sqrt{I} = P_1 \cap P_2$, $P_1P_2 \subseteq I$ and $\sqrt{I}^2 \subseteq I$ where P_1 and P_2 are only distinct prime ideals of S, which are minimal over I.

Proof . By Theorem 2.9, we conclude that either $\sqrt{I} = P$ is a prime ideal of S or $\sqrt{I} = P_1 \cap P_2$, where P_1, and P_2 are only distinct prime ideals of S which are minimal over I. Suppose that $\sqrt{I} = P$ is a prime ideal of S. We are going to show that in both cases $\sqrt{I}^2 \subseteq I$. Let $z \in \sqrt{I}^2$. Then there exist $x, y \in \sqrt{I}$ such that $z = xy$. By Lemma 2.10, $x^2, y^2 \in I$. Therefore $x^2S \subseteq I$ and $y^2S \subseteq I$. We have $xS(xS \cup yS)yS \subseteq I$, and consequently either $xS(xS \cup yS) \subseteq I$ or $(xS \cup yS)yS \subseteq I$ or $xSyS \subseteq I$, since I is a strongly 2-absorbing ideal of S. We conclude $z = xy \in I$. Thus $\sqrt{I}^2 \subseteq I$. Now, suppose that $\sqrt{I} = P_1 \cap P_2$, where P_1, P_2 are only distinct prime ideals of S which are minimal over I. Let $z \in \sqrt{I}^2$. Then there exist $x, y \in \sqrt{I}$ such that $z = xy$. By the same argument which is given above, $x, y \in I$ which implies that $\sqrt{I}^2 \subseteq I$. Now we show that $P_1P_2 \subseteq I$. We have four cases.
Case (1) Let $z = xy \in P_1P_2$ for some $x \in P_1 \setminus P_2$ and $y \in P_2 \setminus P_1$. Therefore $xS \subseteq P_1$, $xS \nsubseteq P_2$, $yS \subseteq P_2$, and $yS \nsubseteq P_1$. With the same argument of Theorem 2.9, we have $xSyS \subseteq I$ and $z = xy \in I$.
Case (2) Let $z = xy \in P_1P_2$ for some $x, y \in P_1 \cap P_2 = \sqrt{I}$. Since $\sqrt{I}^2 \subseteq I$, then $z = xy \in I$.
Case (3) Let $z = xy \in P_1P_2$ for some $x \in P_1 \cap P_2$ and $y \in P_2 \setminus P_1$, and $t \in P_1 \setminus P_2$. Then $(xS \cup tS) \subseteq P_1$, $(xS \cup tS) \nsubseteq P_2$, $yS \subseteq P_2$ and $yS \nsubseteq P_1$. By the same argument as in the proof of Theorem 2.9, we have $(xS \cup tS)(yS) \subseteq I$. This implies that $(xS)(yS) \subseteq I$, and so we have $z = xy \in I$.
Case (4) A similar argument of case (3) shows that if $y \in \sqrt{I} = P_1 \cap P_2$ and $x \in P_1 \setminus P_2$, then $z = xy \in I$, and so $P_1P_2 \subseteq I$. □

3 Basic properties of 2-absorbing subacts

In this section, we define and study 2-absorbing and strongly 2-absorbing acts and subacts over monoids with unique zero.

Definition 3.1. A proper subact B of an S-act A is called *2-absorbing* whenever $aSsSs' \subseteq B$ for $s, s' \in S$ and $a \in A$, then either $aSs \subseteq B$ or $aSs' \subseteq B$ or $sSs' \subseteq (B : A)$.
A centered S-act A is called a *2-absorbing act*, whenever $\Theta = \{\theta\}$is a 2-absorbing subact of A.

Definition 3.2. A proper subact B of A is called *strongly 2-absorbing* whenever the inclusion $CIJ \subseteq B$ for any right ideals $I, J \subseteq S$ and any subact C of A, implies that either $I \subseteq (B : C)$ or $J \subseteq (B : C)$ or $IJ \subseteq (B : A)$.

Example 3.3. (1) Consider $B_{\mathbb{Z}} = \{(z, z) | z \in \mathbb{Z}\}$ as a subact of $(\mathbb{Z} \times \mathbb{Z})_{\mathbb{Z}}$. If $(m, n)\mathbb{Z}z_1\mathbb{Z}z_2 \subseteq B$, for every $z_1, z_2 \in \mathbb{Z}$ and $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, then $m = n$ or $z_1z_2 = 0$. Since $m = n$, then $(m, n)\mathbb{Z}z_1 \subseteq B$ and $(m, n)\mathbb{Z}z_2 \subseteq B$. If $z_1z_2 = 0$, then $z_1\mathbb{Z}z_2 \subseteq (B_{\mathbb{Z}} : (\mathbb{Z} \times \mathbb{Z})_{\mathbb{Z}})$ since $(B_{\mathbb{Z}} : (\mathbb{Z} \times \mathbb{Z})_{\mathbb{Z}}) = 0$. Therefore $B_{\mathbb{Z}}$ is a 2-absorbing subact of $(\mathbb{Z} \times \mathbb{Z})_{\mathbb{Z}}$.
(2) Let S be a left zero semigroup and let $S^1 = S \cup \{1\}$. Any subset I of the monoid S^1 is a 2-absorbing and strongly 2-absorbing ideal. Since $\{s\} = sS^1s'S^1s'' \subseteq I$ for $s, s', s'' \in S$, then $\{s\} = sS^1s' \subseteq I$. Any subact B of an act A over S^1 is a (strongly) 2-absorbing subact, since $aS^1 = aS^1s = aS^1sS^1s' \subseteq B$ for every $s, s' \in S^1$ and $a \in A$.

It is clear that an ideal I of a monoid S is a (strongly) 2-absorbing ideal of S if and only if I_S is a (strongly) 2-absorbing as an S-subact of S_S. Since subacts of S_S are ideals of S and $(J : S) = J$, for each ideal J of S, the assertion follows.

Proposition 3.4. Every nonzero subact B of a 2-absorbing act A is a 2-absorbing act.

Proof . Suppose $bSsSs' = \Theta$ for $s, s' \in S$ and $b \in B$. If $bSs = \Theta$ or $bSs' = \Theta$, then there is nothing to prove. Let $bSs \neq \Theta$ and $bSs' \neq \Theta$. Since A is a 2-absorbing act, then Θ is a 2-absorbing subact of A. It follows that $sSs' \subseteq (\Theta : A) \subseteq (\Theta : B)$. Hence B is a 2-absorbing act. □

Now we investigate relations between prime, 2-absorbing, and strongly 2-absorbing subacts of acts.

Proposition 3.5. Every prime subact of an act is a strongly 2-absorbing subact, and every strongly 2-absorbing subact of an act is a 2-absorbing subact.

Proof . Let B be a prime subact of an S-act A and let $CIJ \subseteq B$ for any ideals I, J of S and any subact C of A. Since B is a prime subact of A, we have $C \subseteq B$ or $IJ \subseteq (B : A)$. Then $CI \subseteq C \subseteq B$ and $CJ \subseteq C \subseteq B$ or $IJ \subseteq (B : A)$. Hence B is a strongly 2-absorbing subact of A.
Let B be a strongly 2-absorbing subact of A and let $aSsSs' \subseteq B$ for $s, s' \in S$ and $a \in A$. Therefore $aSsSs'S \subseteq BS \subseteq B$. It is clear that sS and $s'S$ are ideals of S and that aS is a subact of A. Also since B is a strongly 2-absorbing subact of A, then either $aSsS \subseteq B$ or $aSs'S \subseteq B$ or $sSs'S \subseteq (B : A)$. As $1 \in S$, we can conclude that either $aSs \subseteq B$ or $aSs' \subseteq B$ or $sSs' \subseteq (B : A)$. □

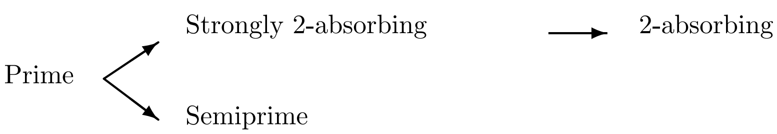
The following example shows that the converse of Proposition 3.5, is not necessarily true.

Example 3.6. Let $(N = \mathbb{N} \cup \{0\}, .)$ be the monoid of nonnegative integer numbers with respect to multiplication. For any prime integers $p, q, r \in N$,
(1) $pN = \{pm : m \in N\}$ is a prime subact of N.
(2) p^2N is a 2-absorbing subact of N, but it is not a semiprime subact of N and then it is not a prime subact. Since $p^2 \in p^2N$, but $p \notin p^2N$.
(3) pqN is a semiprime subact of N but is not a prime subact. Since we have $pq \in pqN$, but $p \notin pqN$ and $q \notin pqN$.
(4) $p^2N \cup q^2N$ is a 2-absorbing subact of N. Although it is not a strongly 2-absorbing subact. Since

$$(pN \cup qN)(pN \cup qN)(pN \cup qN) = p^3N \cup q^3N \cup p^2qN \cup pq^2N \subseteq p^2N \cup q^2N,$$

but $(pN \cup qN)(pN \cup qN) = p^2N \cup q^2N \cup pqN \nsubseteq p^2N \cup q^2N$.
(5) $pqrN$ is a semiprime subact of N, but it is not a 2-absorbing subact and then it is not a strongly 2-absorbing subact too. Note that $pqr \in pqrN$, but $pq \notin pqrN$ and $pr \notin pqrN$ and $qr \notin pqrN$.

Therefore we get the following relations strictly:



Now, we are going to prove some algebraic structures of (strongly) 2-absorbing acts and subacts.

Proposition 3.7. The intersection of every two prime subacts of an S-act is a strongly 2-absorbing subact.

Proof . Let P_1 and P_2 be two prime subacts of an S-act A and let $BIJ \subseteq P_1 \cap P_2$ for a subact B of A and ideals I and J of S. Then $BIJ \subseteq P_1$ and $BIJ \subseteq P_2$. Since P_1 and P_2 are prime subacts of A and by Proposition 3 of [1], $(P_1 : A)$ and $(P_2 : A)$ are prime ideals of S, then either $B \subseteq P_1$ or $I \subseteq (P_1 : A)$ or $J \subseteq (P_1 : A)$ and either $B \subseteq P_2$ or $I \subseteq (P_2 : A)$ or $J \subseteq (P_2 : A)$. In any case, it can be concluded that either $BI \subseteq P_1 \cap P_2$ or $BJ \subseteq P_1 \cap P_2$ or $IJ \subseteq (P_1 \cap P_2 : A)$. Hence $P_1 \cap P_2$ is a strongly 2-absorbing subact of A. □

Proposition 3.8. Let B be a 2-absorbing subact of an S-act A and let $\vartheta \in Con(A)$. Then $B/\vartheta_{\lfloor B}$ is a 2-absorbing subact of A/ϑ.

Proof . For $s_1, s_2 \in S$ and $[a]_\vartheta \in A/\vartheta$, let $[a]_\vartheta Ss_1Ss_2 \subseteq B/\vartheta_{\lfloor B}$. We are going to show that either $[a]_\vartheta Ss_1 \subseteq B/\vartheta_{\lfloor B}$ or $[a]_\vartheta Ss_2 \subseteq B/\vartheta_{\lfloor B}$ or $s_1Ss_2 \subseteq (B/\vartheta_{\lfloor B} : A/\vartheta)$. From $[a]_\vartheta Ss_1Ss_2 \subseteq B/\vartheta_{\lfloor B}$, we conclude $aSs_1Ss_2 \subseteq B$ for $a \in A$. Then either $aSs_1 \subseteq B$ or $aSs_2 \subseteq B$ or $s_1Ss_2 \subseteq (B : A)$, since B is a 2-absorbing subact of A. Therefore either $[a]_\vartheta Ss_1 \subseteq B/\vartheta_{\lfloor B}$ or $[a]_\vartheta Ss_2 \subseteq B/\vartheta_{\lfloor B}$ or $s_1Ss_2 \subseteq (B/\vartheta_{\lfloor B} : A/\vartheta)$. □

Corollary 3.9. Let B be a 2-absorbing subact of an S-act A, let $\vartheta \in Con(A)$, and let $B^{\vartheta} = \{a \in A \mid B \cap [a]_{\vartheta} \neq \emptyset\ \}$. Then $B^{\vartheta}/\vartheta_{\restriction_{B^{\vartheta}}}$ is a 2-absorbing subact of A/ϑ.

Proof . By Proposition 3.8, $B/\vartheta_{\restriction B}$ is a 2-absorbing subact of A/ϑ, and by the third isomorphism theorem of algebra (see [3, Theorem 6.18]), we have $B/\vartheta_{\restriction_B} \cong B^{\vartheta}/\vartheta_{\restriction_{B^{\vartheta}}}$. Hence $B^{\vartheta}/\vartheta_{\restriction_{B^{\vartheta}}}$ is a 2-absorbing subact of A/ϑ. □

The concepts of congruence and quotient acts closely related, then we can have the following result.

Corollary 3.10. If C is a subact of B and B is a 2-absorbing subact of A, then the Rees factor B/C is a 2-absorbing subact of A/C.

Proof . We know for the Rees congruence $\rho_C \in Con(A)$, that is, for $a, a' \in A$, $a\rho_C a'$ if and only if $a = a'$ or $a, a' \in C$, we have A/ρ_C equals the Rees factor A/C. By Proposition 3.8, the Rees factor B/C is a 2-absorbing subact of A/C. □

Theorem 3.11. (1) If B is a strongly 2-absorbing subact of an act A over monoid S, then $(B : A)$ is a strongly 2-absorbing ideal of the monoid S.

(2) Let B be a proper subact of a cyclic act A over a commutative monoid S. If $(B : A)$ is a strongly 2-absorbing ideal of S, then B is a strongly 2-absorbing subact of A.

Proof . (1) Let I, J, K be three ideals of S such that $IJK \subseteq (B : A)$; then $AIJK = (AI)JK \subseteq B$. Since B is a strongly 2-absorbing subact of A, we have either $AIJ \subseteq B$ or $AJK \subseteq B$ or $JK \subseteq (A : B)$. Hence $IJ \subseteq (A : B)$ or $JK \subseteq (A : B)$ or $JK \subseteq (A : B)$. Therefore $(A : B)$ is a strongly 2-absorbing ideal of S.

(2) Let $A = aS$ be a cyclic act and let $(B : A)$ be a strongly 2-absorbing ideal of S. Consider $CIJ \subseteq B$, for the subact C of A and ideals I and J of S. If $CI \nsubseteq B$ and $CJ \nsubseteq B$, then there exist $as \in C \leq aS$ and $at \in C \leq aS$ for some $s, t \in S$ such that $(as)I \nsubseteq B$ and $(at)J \nsubseteq B$. But we have $(as)IJ \subseteq B$ and $(at)IJ \subseteq B$, since $CIJ \subseteq B$. We have $a(sS \cup tS)IJ \subseteq B$. Therefore $aS(sS \cup tS)IJ \subseteq BS \subseteq B$ and $(sS \cup tS)IJ \subseteq (B : A)$, because S is a commutative monoid. Since $(B : A)$ is a strongly 2-absorbing ideal, either $(sS \cup tS)I \subseteq (B : A)$ or $(sS \cup tS)J \subseteq (B : A)$ or $IJ \subseteq (B : A)$. Thus we just have $IJ \subseteq (B : A)$, since $(as)I \nsubseteq B$ and $(at)J \nsubseteq B$, then $(sS \cup tS)I \nsubseteq (B : A)$ or $(sS \cup tS)J \nsubseteq (B : A)$. Hence B is a strongly 2-absorbing subact of A. □

In Theorem 3.11, we showed that the converse of first statement is true for cyclic acts over commutative monoids. In the next proposition, we are going to characterize strongly 2-absorbing acts and monoids.

Proposition 3.12. A monoid S is strongly 2-absorbing if and only if there exists a strongly 2-absorbing faithful S-act.

Proof . If S is a strongly 2-absorbing monoid. Then S_S is a faithful strongly 2-absorbing S-act.
Conversely, suppose that there exists a faithful strongly 2-absorbing S-act A. We show that S is a strongly 2-absorbing monoid or equivalently, 0 is a strongly 2-absorbing ideal of S. Suppose $IJK = 0$ for ideals $I, J,$ and K of S. If $IJ \neq 0 = (\Theta : A)$ and $IK \neq 0 = (\Theta : A)$, then $AIJ \neq \Theta$ and $AIK \neq \Theta$. Therefore there exist $a, b \in A$ such that $aIJ \neq \Theta$ and $bIK \neq \Theta$, and then $\{a, b\}IJ \neq \Theta$ and $\{a, b\}IK \neq \Theta$. Since $AIJK = \Theta$, we have $(\{a, b\}I)JK = \Theta$. Hence A is strongly 2-absorbing, $JK = (\Theta : A) = 0$, that is, S is a strongly 2-absorbing monoid. □

We close this paper by showing the relation between strongly 2-absorbing ideals and acts.

Proposition 3.13. Let I be an ideal of a monoid S. Then I is a strongly 2-absorbing ideal of S if and only if there exists a strongly 2-absorbing S-act A with $(\Theta : A) = I$.

Proof . (1) ⇒ (2). If I is a strongly 2-absorbing ideal of S, then the Rees factor S/I is a strongly 2-absorbing monoid. By Proposition 3.12, there exists a strongly 2-absorbing faithful S/I-act $A_{S/I}$ such that

$$(\Theta : A) = \{[s] \in S/I \mid A[s] = \Theta\} = I.$$

(2) ⇒ (1). If A is a strongly 2-absorbing S-act and $I = (\Theta : A)$, then $A_{S/I}$ is a strongly 2-absorbing S-act such that $[0] = I = (\Theta : A_{S/I})$. By Proposition 3.12, S/I is a strongly 2-absorbing monoid which is means that I is a strongly 2-absorbing ideal of S, as desired. □

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