Journal of Frame and Matrix Theory (JFMT) 1 (2024) No. 2, 1–21
Online ISSN: 3041-8526
Published by Faculty of Mathematics and Computer Science
Hakim Sabzevari University
http://dx.doi.org/10.22034/jfmt.2023.363036.1000

Rings of real measurable functions vanishing at infinity on a measurable space

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(Communicated by Mostafa Abedi)

Abstract

Let $M(X,\mathscr{A})$ be the ring of all real measurable functions on a measurable space $(X,\mathscr{A})$. We show that for every measurable space $(X,\mathscr{A})$, there exists a T-measurable space $(Y,\mathscr{A}')$ such that $M_K(X,\mathscr{A}) \cong M_K(Y,\mathscr{A}')$ and $M_\infty(X,\mathscr{A}) \cong M_\infty(Y,\mathscr{A}')$, where $M_K(X,\mathscr{A})$ is the ring of real measurable functions $f \in M(X,\mathscr{A})$ for which $coz(f)$ is a compact element of $\mathscr{A}$, and $M_\infty(X,\mathscr{A})$ is the ring of real measurable functions vanishing at infinity on $(X,\mathscr{A})$. Then, we introduce σ-compact and locally compact measurable spaces. We prove that a T-measurable space $(X,\mathscr{A})$ is compact (σ-compact) if and only if the set X is finite (at most countable) and $\mathscr{A} = \mathcal{P}(X)$. Next, we obtain several equivalent conditions for $M_\infty(X,\mathscr{A})$ to be a regular ring. Finally, we show that if $(X,\mathscr{A})$ is a T-measurable space and $M_\infty(X,\mathscr{A}) \neq \{0\}$, then

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Received: 11 May 2023 *Accepted:* 10 November 2023

there exists a locally compact measurable space $(Y, \mathscr{A}')$ such that $M_\infty(X, \mathscr{A}) \cong M_\infty(Y, \mathscr{A}')$ and $M_K(X, \mathscr{A}) \cong M_K(Y, \mathscr{A}')$.

Keywords: Real measurable function, Compact measurable space, σ-compact measurable space, Locally compact measurable space, Regular ring.
MSC 2020: Primary: 28A20; Secondary: 13A15, 54C30.

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1 Introduction

The mathematical notion of *measure* represents some of the concepts which are of use in the physical world, such as length, area, volume, mass and electric charge. Measurable spaces have been studied by many authors. By a *σ-algebra* on a non-empty set X we mean a non-empty subset $\mathscr{A}$ of $\mathcal{P}(X)$ which is closed under countable unions and complement. A pair $(X, \mathscr{A})$ is called a *measurable space* if $\mathscr{A}$ is a σ-algebra on X. Let $(X, \mathscr{A})$ be a measurable space and $M(X, \mathscr{A})$, abbreviated as $M(X, \mathscr{A})$, be the set of all $f \in \mathbb{R}^X$ such that for every open subset U of $\mathbb{R}$, $f^{-1}(U) \in \mathscr{A}$. Then, $M(X, \mathscr{A})$ is a subring of $\mathbb{R}^X$, when equipped with the pointwise operations of addition and multiplication.

As Viertl proved in [21], when X is a topological space and $\mathscr{A}$ is the set of all Borel subsets of X, every maximal ideal of $M(X, \mathscr{A})$ is real if and only if $\mathscr{A}$ contains only a finite number of elements, if and only if every ideal is fixed. Another result, established by Hager in [16], says that if $(X, \mathscr{A})$ is a measurable space, then $M(X, \mathscr{A})$ is a regular ring in the sense of von Neumann. Moreover, as shown by Azadi et al. in [5], $M(X, \mathscr{A})$ is an $\aleph_0$-self-injective ring for any measurable space $(X, \mathscr{A})$.

In [4], Amini et al. simultaneously generalized the ring of real-valued continuous functions and the ring of real-valued measurable functions. Also, Momtahan in [19] studied essential ideals, socle, and some related ideals of rings of real-valued measurable functions. He also studied the Goldie dimension of rings of measurable functions.

Throughout the paper, $C(X)$ stands for the set of all real-valued continuous functions defined on X, with the pointwise operations of addition and multiplication (see [17] and [14]). For any completely regular Hausdorff space X, $C_\infty(X)$, the subring of all functions in $C(X)$ which vanish at infinity, was introduced by Kohls in [18] (also, see [2], [7] and [6]). A result obtained by A.R. Aliabad et al. (2004) says that for any completely regular Hausdorff space X, if $C_\infty(X) \neq \{0\}$, then there exists a locally compact space Y such that $C_\infty(X) \cong C_\infty(Y)$ (see [3]).

As usual, let $\mathcal{R}L$ denote the ring of real-valued continuous functions on a completely regular frame L (see [8] and [9]). $\mathcal{R}_\infty L$, the ring of real continuous functions vanishing at infinity on a frame L, was first discussed by Dube [10] (also, see [1] and [15]). In [12], Estaji et al. proved, under some conditions, the existence of a locally compact frame M such that $\mathcal{R}_\infty(L) \cong \mathcal{R}_\infty(M)$.

The present paper is devoted to placing these results in a measurable space context. It is organized as follows.

Section 2 presents the prerequisites needed for the rest of the paper. The definitions and results of this section are taken from [20] and [13].

In Section 3, we prove that if $(X, \mathscr{A})$ is a T-measurable space, then the measurable space $(X, \mathscr{A})$ is a compact (σ-compact) measurable space if and only if the set X is a finite (at most countable) set and $\mathscr{A} = \mathcal{P}(X)$ (see Propositions 3.2 and 3.6). Next, for every T-measurable space $(X, \mathscr{A})$, we show that $(X, \mathscr{A})$ is a compact T-measurable space if and only if $M(X, \mathscr{A}) = M^*(X, \mathscr{A})$, where $M^*(X, \mathscr{A})$ is the set of all bounded elements of $M(X, \mathscr{A})$ (see Proposition 3.7).

In Section 4, we prove that for every measurable space $(X, \mathscr{A})$, there exists a T-measurable space $(Y, \mathscr{A}')$ such that $M_\infty(Y, \mathscr{A}') \cong M_\infty(X, \mathscr{A})$ (see Proposition 4.3).

In Section 5, we show that for every measurable space $(X, \mathscr{A})$, the ring $M_\infty(X, \mathscr{A})$ is an ideal of $M^*(X, \mathscr{A})$ (see Proposition 5.4). Also, we obtain several equivalent conditions for $M_\infty(X, \mathscr{A})$ to be an ideal of $M(X, \mathscr{A})$ (see Proposition 5.9). Next, we prove that if $(X, \mathscr{A})$ is a T-measurable space, then $M_K(X, \mathscr{A})$ is a regular ring (see Proposition 5.12). Also, we obtain several equivalent conditions for $M_\infty(X, \mathscr{A})$ to be a regular ring (see Proposition 5.14).

Finally, in Section 6, we obtain for every T-measurable space $(X, \mathscr{A})$ several equivalent conditions for $M_\infty(X, \mathscr{A})$ to be a free ideal of $M^*(X, \mathscr{A})$ (see Proposition 6.4). Also, we show that when $(X, \mathscr{A})$ is a T-measurable space and $M_\infty(X, \mathscr{A}) \neq \{0\}$, there exists a locally compact measurable space $(Y, \mathscr{A}')$ such that $M_\infty(X, \mathscr{A}) \cong M_\infty(Y, \mathscr{A}')$ and $M_K(X, \mathscr{A}) \cong M_K(Y, \mathscr{A}')$ (see Proposition 6.6).

2 Preliminaries

We recall from [20, Theorem 1.10] that if $\mathcal{A}$ is any collection of subsets of X, there exists a smallest σ-algebra $\mathscr{A}^*$ in X such that $\mathcal{A} \subseteq \mathscr{A}^*$. This is called the *$\sigma$-algebra generated by $\mathcal{A}$*, and denoted by $< \mathcal{A} >$. Since the intersection of any family of σ-algebras in X is a σ-algebra in X, we conclude that $< \mathcal{A} >$ is the intersection of all σ-algebras in X which contain $\mathcal{A}$. Hence, $< \mathcal{A} >=< \mathcal{A}^c >=< \mathcal{A} \cup \mathcal{A}^c >$, where $\mathcal{A}^c := \{A^c : A \in \mathcal{A}\}$. Also, if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ with $\mathcal{A} \subseteq \mathcal{B}$, then $< \mathcal{A} >\subseteq< \mathcal{B} >$.

Given $f \in \mathbb{R}^X$, the set $f^{-1}(0)$ will be called the *zero-set* of f. We shall find it convenient to denote this set by $Z(f)$, or, for clarity, by $Z_X(f)$:

$$Z(f) = Z_X(f) = \{x \in X : f(x) = 0\}.$$

For any subset A of $M(X, \mathscr{A})$, we write $Z_{\mathscr{A}}[A] = \{Z(f) : f \in A\}$ and $Z_{\mathscr{A}}[X] = Z_{\mathscr{A}}[M(X, \mathscr{A})]$. Also, the set $X \setminus Z(f)$, denoted by $\operatorname{coz}(f)$, will be called the *cozero-set* of f.

Proposition 2.1. [13] If $(X, \mathscr{A})$ is a measurable space, then $Z_{\mathscr{A}}[X] = \mathscr{A}$.

Definition 2.2. [13] Let I be any ideal in $M(X, \mathscr{A})$. If $\bigcap_{f \in I} Z(f)$ is non-empty, we call I a *fixed ideal*; if $\bigcap_{f \in I} Z(f) = \emptyset$, then I is a *free ideal*. Also, if $\mathcal{K} \subseteq \mathscr{A}$ with $\bigcap \mathcal{K}$ non-empty, we call $\mathcal{K}$ a *fixed subset* of $\mathscr{A}$; if $\bigcap \mathcal{K} = \emptyset$, then $\mathcal{K}$ is a *free subset* of $\mathscr{A}$.

Let $(X, \mathscr{A})$ be a measurable space. Throughout this paper we let

$$M_x := \{f \in M(X, \mathscr{A}) : f(x) = 0\},$$

for every $x \in X$. It is evident that M_x is a fixed ideal of $M(X, \mathscr{A})$ for every $x \in X$.

An element a of a bounded lattice L is said to be *compact* if $a = \bigvee S$, $S \subseteq L$, implies $a = \bigvee T$ for some finite subset T of S. A bounded lattice L is said to be *compact* whenever its top element $\top$ is compact (see [11]). A measurable space $(X, \mathscr{A})$ is called a *compact measurable space* if $\mathscr{A}$ is a compact lattice.

Proposition 2.3. [13] Let $(X, \mathscr{A})$ be a measurable space. Then, the following statements are equivalent.

1. $(X, \mathscr{A})$ is compact.
2. Every proper ideal in $M(X, \mathscr{A})$ is fixed.
3. Every maximal ideal in $M(X, \mathscr{A})$ is fixed. In fact, for every maximal ideal M of $M(X, \mathscr{A})$, there exists an element x of X such that $M = M_x$.

Definition 2.4. [13] A measurable space $(X, \mathscr{A})$ is said to be *T-measurable* if, whenever x and y are distinct points in X, there exists a measurable set containing one and not the other.

Let $(X, \mathscr{A})$ be a measurable space and $Y \subseteq X$. Then $(Y, \mathscr{A}_Y)$ is a measurable space, where $\mathscr{A}_Y = \{Y \cap A : A \in \mathscr{A}\}$. This notation will be used throughout the paper. If $(X, \mathscr{A})$ is a T-measurable space, then so is $(Y, \mathscr{A}_Y)$.

Proposition 2.5. [13] Let $(X, \mathscr{A})$ be a measurable space. Then, the following statements are equivalent.

(1) The measurable space $(X, \mathscr{A})$ is a T-measurable space.
(2) If x and y are distinct points in X, then $M_x \neq M_y$.
(3) For every maximal ideal M in $M(X, \mathscr{A})$, $|\bigcap_{f \in M} Z(f)| \leq 1$.

Corollary 2.6. [13] For every compact measurable space $(X, \mathscr{A})$, there exists a compact T-measurable space $(Y, \mathscr{A}')$ such that $M(X, \mathscr{A}) \cong M(Y, \mathscr{A}')$, as rings.

3 Compact measurable spaces

In this section, we characterize compact measurable spaces and σ-compact measurable spaces. We begin with the following lemma.

Lemma 3.1. If $(X, \mathscr{A})$ is an infinite T-measurable space, then there exists an infinite countable subset $\{B_n\}_{n \in \mathbb{N}}$ of $\mathscr{A} \setminus \{\emptyset\}$ such that $X = \bigcup_{n \in \mathbb{N}} B_n$ and $B_n \cap B_m = \emptyset$ for every $n, m \in \mathbb{N}$ with $n \neq m$.

Proof . Consider $\{x_n : n \in \mathbb{N}\} \subseteq X$ with distinct elements. For every $n, m \in \mathbb{N}$ with $n \neq m$, there exists an element D_{nm} in $\mathscr{A}$ such that $x_n \in D_{nm}$ and $x_m \notin D_{nm}$. Then, $x_n \in D_n := \bigcap_{m \in \mathbb{N}} D_{nm}$ and $\{x_m : \ m \in \mathbb{N} \text{ with } n \neq m\} \cap D_n = \emptyset$ for every $n \in \mathbb{N}$. Now, we let $B_n := D_n \setminus \bigcup_{\substack{n \neq m \in \mathbb{N} \\ m \neq n}} D_m$ for any $n \in \mathbb{N}$. If $X = \bigcup_{n \in \mathbb{N}} B_n$, this completes the proof. Otherwise, we let $B_0 := X \setminus \bigcup_{n \in \mathbb{N}} B_n$. Then $\{B_n\}_{n \in \mathbb{N} \cup \{0\}} \subseteq \mathscr{A} \setminus \{\emptyset\}$, $X = \bigcup_{n \in \mathbb{N} \cup \{0\}} B_n$ and $B_n \cap B_m = \emptyset$ for every $n, m \in \mathbb{N} \cup \{0\}$ with $n \neq m$. $\square$

In the following proposition, a compact measurable space $(X, \mathscr{A})$ is characterized in terms of the cardinality of X.

Proposition 3.2. Let $(X, \mathscr{A})$ be a T-measurable space. Then, the measurable space $(X, \mathscr{A})$ is a compact measurable space if and only if the set X is a finite set and $\mathscr{A} = \mathcal{P}(X)$.

Proof . *Necessity.* Let $(X, \mathscr{A})$ be a compact measurable space. Assume, to the contrary, that X is not finite. By Lemma 3.1, there exists an infinite countable subset $\{B_n\}_{n \in \mathbb{N}}$ of $\mathscr{A} \setminus \{\emptyset\}$ such that $X = \bigcup_{n \in \mathbb{N}} B_n$ and $B_n \cap B_m = \emptyset$, for every $n, m \in \mathbb{N}$ with $n \neq m$. This contradicts the fact that $(X, \mathscr{A})$ is a compact measurable space. Hence, X is a finite set. We assume that $X = \{x_1, x_2, \ldots, x_k\}$, where $k \in \mathbb{N}$. For every $1 \leq n, m \leq k$ with $n \neq m$, there exists $D_{nm} \in \mathscr{A}$ such that $x_n \in D_{nm}$ and $x_m \notin D_{nm}$. Then, $\{x_n\} = \bigcap_{\substack{1 \leq m \leq k \\ n \neq m}} D_{nm} \in \mathscr{A}$. Hence, $\mathscr{A} = \mathcal{P}(X)$.

Sufficiency. This is evident. $\square$

As an immediate consequence, we obtain the following corollary.

Corollary 3.3. If $(X, \mathscr{A})$ is a compact measurable space, then $M(X, \mathscr{A}) \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$.

Proof . By Corollary 2.6, there exists a compact T-measurable space $(Y, \mathscr{A}')$ such that $M(X, \mathscr{A}) \cong M(Y, \mathscr{A}')$ as rings. Hence, by Proposition 3.2, there exists an element n in $\mathbb{N}$ such that

$$M(X, \mathscr{A}) \cong M(Y, \mathscr{A}') \cong \mathbb{R}^Y \cong \mathbb{R}^n.$$

□

Definition 3.4. A measurable space $(X, \mathscr{A})$ is *σ-compact* provided that there exists a subset $\{A_n\}_{n\in\mathbb{N}}$ of $\mathscr{A}$ such that for every $n \in \mathbb{N}$, A_n is a compact element of $\mathscr{A}$, and $X = \bigcup_{n\in\mathbb{N}} A_n$.

Remark 3.5. If $(X, \mathscr{A})$ is a σ-compact measurable space, then there exists a collection $\{A_n\}_{n\in\mathbb{N}}$ of $\mathscr{A}$ with $A_n \subseteq A_{n+1}$ for $n \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, A_n is a compact element of $\mathscr{A}$, and $X = \bigcup_{n\in\mathbb{N}} A_n$.

Now, we are able to characterize a σ-compact measurable space $(X, \mathscr{A})$ in terms of the cardinality of X.

Proposition 3.6. Let $(X, \mathscr{A})$ be a T-measurable space. Then, $(X, \mathscr{A})$ is a σ-compact measurable space if and only if $|X| \leq \aleph_0$ and $\mathscr{A} = \mathcal{P}(X)$.

Proof . *Necessity.* Let $\{A_n : n \in \mathbb{N}\}$ be a family of compact elements of $\mathscr{A}$ such that $A = \bigcup_{n\in\mathbb{N}} A_n$. Since $(A_n, \mathscr{A}_{A_n})$ is a compact T measurable space, we conclude from Proposition 3.2 that A_n is a finite set and $\mathscr{A}_{A_n} = \mathcal{P}(A_n)$ for every $n \in \mathbb{N}$. So, $|X| \leq \aleph_0$.

Consider $a \in X$. Then, there exists an element n in $\mathbb{N}$ such that $a \in A_n$. By Proposition 3.2, we deduce from $\mathscr{A}_{A_n} = \mathcal{P}(A_n)$ that there exists an element A of $\mathscr{A}$ such that $\{a\} = A \cap A_n \in \mathscr{A}$. Therefore, $\mathscr{A} = \mathcal{P}(X)$.

Sufficiency. This is evident. □

Let $(X, \mathscr{A})$ be a T-measurable space. We consider the bounded part $M^*(X, \mathscr{A})$ of $M(X, \mathscr{A})$ consisting of all $f \in M(X, \mathscr{A})$ such that there exists a natural number n with $|f(x)| < n$ for every $x \in X$. The subset $M^*(X, \mathscr{A})$ of $M(X, \mathscr{A})$ is also closed under the algebraic and order operations of $M(X, \mathscr{A})$. Therefore, $M^*(X, \mathscr{A})$ is a sub-f-ring of $M(X, \mathscr{A})$.

The next result is a new characterization of compact T-measurable spaces in terms of the bounded part $M^*(X, \mathscr{A})$ of $M(X, \mathscr{A})$.

Proposition 3.7. Let $(X, \mathscr{A})$ be a T-measurable space. Then, $M(X, \mathscr{A}) = M^*(X, \mathscr{A})$ if and only if $(X, \mathscr{A})$ is a compact T-measurable space.

Proof . *Necessity.* Suppose that $M(X, \mathscr{A}) = M^*(X, \mathscr{A})$, and that X is an infinite set. Then, by Lemma 3.1, there exists an infinite countable subset $\{B_n\}_{n\in\mathbb{N}}$ of $\mathscr{A} \setminus \{\emptyset\}$ such that $X = \bigcup_{n\in\mathbb{N}} B_n$ and $B_n \cap B_m = \emptyset$ for every $n, m \in \mathbb{N}$ with $n \neq m$. We define $g : X \to \mathbb{R}$ by $g(x) = n$ if $x \in B_n$. It is clear that $g \in M(X, \mathscr{A}) \setminus M^*(X, \mathscr{A})$, which is a contradiction. Hence, by Proposition 3.2, $(X, \mathscr{A})$ is a compact T-measurable space.

Sufficiency. Consider $f \in M(X, \mathscr{A})$. Since

$$X = \bigcup_{n\in\mathbb{N}} f^{-1}(-n, n) \text{ and } \{f^{-1}(-n, n)\}_{n\in\mathbb{N}} \subseteq \mathscr{A},$$

we conclude that there exists a natural number n such that $f^{-1}(-n, n) = X$. Hence, $M(X, \mathscr{A}) = M^*(X, \mathscr{A})$. □

4 On the subring of all functions in $M(X, \mathscr{A})$ which vanish at infinity

In this section, we show that when an algebraic or lattice property holds for $M_\infty(X, \mathscr{A})$ with $(X, \mathscr{A})$ being a T-measurable space, it also holds for $M_\infty(X, \mathscr{A})$ with arbitrary $(X, \mathscr{A})$. To begin with, we need to define $M_\infty(X, \mathscr{A})$.

Definition 4.1. Let $M_\infty(X, \mathscr{A})$ denote the family of all functions $f \in M(X, \mathscr{A})$ for which the set $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is a compact element of $\mathscr{A}$ for every $n \in \mathbb{N}$.

Proposition 4.2. If $(X, \mathscr{A})$ is a measurable space, then $M_\infty(X, \mathscr{A})$ is a subring of $M(X, \mathscr{A})$.

Proof . Consider $f, g \in M_\infty(X, \mathscr{A})$. Since

$$\left\{x \in X : |f(x) + g(x)| \geq \frac{1}{n}\right\} \subseteq \left\{x \in X : |f(x)| \geq \frac{1}{2n}\right\} \cup \left\{x \in X : |g(x)| \geq \frac{1}{2n}\right\},$$

we find that $\{x \in X : |f(x) + g(x)| \geq \dfrac{1}{n}\}$ is a compact element of $\mathscr{A}$ for every $n \in \mathbb{N}$. Hence, $f + g \in M_\infty(X, \mathscr{A})$. Also, we deduce from

$$\left\{x \in X : |f(x) \times g(x)| \geq \frac{1}{n}\right\} \subseteq \left\{x \in X : |f(x)| \geq \frac{1}{[\sqrt{n}] + 1}\right\} \cup \left\{x \in X : |g(x)| \geq \frac{1}{[\sqrt{n}] + 1}\right\},$$

that $\{x \in X : |f(x)g(x)| \geq \dfrac{1}{n}\}$ is a compact element of $\mathscr{A}$ for every $n \in \mathbb{N}$. Thus, $fg \in M_\infty(X, \mathscr{A})$. Therefore, $M_\infty(X, \mathscr{A})$ is a subring of $M(X, \mathscr{A})$. □

The next proposition shows that there is no need to consider rings of real measurable functions other than T-measurable spaces.

Proposition 4.3. For every measurable space $(X, \mathscr{A})$, there exists a T-measurable space $(Y, \mathscr{A}')$ such that

$$M_\infty(Y, \mathscr{A}') \cong M_\infty(X, \mathscr{A}).$$

Proof . Define $x \sim x'$ in X to mean that $f(x) = f(x')$ for every $f \in M(X, \mathscr{A})$. For each $f \in M(X, \mathscr{A}))$, define a function $h_f \in \mathbb{R}^{X/\sim}$ by $h_f([x]_\sim) = f(x)$. Let $\mathscr{A}'$ be the weak measurable space induced by $\{h_f : f \in M(X, \mathscr{A})\}$ on $X/\sim$. Consider $Y := X/\sim$. Then, $(Y, \mathscr{A}')$ is a T–measurable space. Define $\theta : X \to Y$ by $\theta(x) = [x]_\sim$. The function θ is onto, and $h_f \circ \theta = f$ for any $f \in M(X, \mathscr{A})$. We define $\eta : M(Y) \to M(X, \mathscr{A})$ by $g \mapsto g \circ \theta$. Then, η is an isomorphism.

Consider $g \in M_\infty(Y, \mathscr{A}')$ and $n \in \mathbb{N}$. We show that $A_n := \{x \in X : |\eta(g)(x)| \geq \frac{1}{n}\}$ is a compact element of $\mathscr{A}$. Since

$$A_n = \left\{x \in X : |g([x])| \geq \frac{1}{n}\right\} \text{ and } \theta(A_n) = \left\{[x]_\sim \in Y : |g([x])| \geq \frac{1}{n}\right\}$$

is a compact element of $\mathscr{A}'$, we conclude that $\theta^{-1}\theta(A_n)$ is a compact element of $\mathscr{A}$. Since $A_n \in\downarrow \theta^{-1}\theta(A_n)$, we deduce that A_n is a compact element of $\mathscr{A}$. Therefore, $\eta(g) \in M_\infty(X, \mathscr{A})$.

Consider $f \in M_\infty(X, \mathscr{A})$ and $n \in \mathbb{N}$. We show that

$$B_n := \left\{[x]_\sim \in Y : |\eta^{-1}(f)([x])| \geq \frac{1}{n}\right\}$$

is a compact element of $\mathscr{A}'$. That $\theta^{-1}(B_n) = \{x \in X : |f^{-1}(x)| \geq \frac{1}{n}\}$ is a compact element of $\mathscr{A}$ allows us to conclude that $B_n = \theta(\theta^{-1}(B_n))$ is a compact element of $\mathscr{A}'$. Hence, $\eta^{-1}(f) \in M_\infty(Y, \mathscr{A}')$. Accordingly, $\eta|_{M_\infty(Y,\mathscr{A}')} : M_\infty(Y, \mathscr{A}') \to M_\infty(X, \mathscr{A})$ is an isomorphism. □

By Proposition 4.3, to study $M_\infty(X, \mathscr{A})$, we can assume that X is a T-measurable space.

Proposition 4.4. Let $(X, \mathscr{A})$ be a T-measurable space and $A \in \mathscr{A}$. Then, $(A, \mathscr{A}_A)$ is a σ-compact measurable space if and only if there exists an element f in $M_\infty(X, \mathscr{A})$ such that $\operatorname{coz}(f) = A$.

Proof . *Necessity.* Let $\{A_n : n \in \mathbb{N}\}$ be a family of compact elements of $\mathscr{A}_A$ such that $A = \bigcup_{n\in\mathbb{N}} A_n$ and $A_n \subseteq A_{n+1}$ for every $n \in \mathbb{N}$. We let $B_1 := A_1$ and $B_n = A_n \setminus A_{n-1}$ for every $2 \leq n \in \mathbb{N}$. It is clear that B_n is a compact element of $\mathscr{A}$ for every $n \in \mathbb{N}$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x \in B_n \\ 0 & \text{otherwise,} \end{cases}$$

for every $x \in X$. From

$$f^{-1}([r,\infty)) = \begin{cases} \emptyset & \text{if } r > 1 \\ X & \text{if } r \leq 0 \\ \bigcup_{\substack{m\in\mathbb{N} \\ m\leq n}} B_m & \text{if } \frac{1}{n+1} < r \leq \frac{1}{n} \text{ for some } n \in \mathbb{N}, \end{cases}$$

we infer that $f \in M(X, \mathscr{A})$ and $\operatorname{coz}(f) = A$. Since

$$\left\{x \in X : |f(x)| \geq \frac{1}{n}\right\} = \left\{x \in \bigcup_{m\in\mathbb{N}} B_m : |f(x)| \geq \frac{1}{n}\right\} = \bigcup_{m\leq n} B_m,$$

we conclude that $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is a compact element of $\mathscr{A}$ for every $n \in \mathbb{N}$. Therefore, $f \in M_\infty(X, \mathscr{A})$.

Sufficiency. Consider $f \in M_\infty(X, \mathscr{A})$ and $A = \operatorname{coz}(f)$. Then,

$$A_n := \left\{x \in X : |f(x)| \geq \frac{1}{n}\right\}$$

is a compact element of $\mathscr{A}$, for every $n \in \mathbb{N}$. From $A_n \subseteq \operatorname{coz}(f)$ we conclude that A_n is a compact element of $\mathscr{A}_A$ for every $n \in \mathbb{N}$. Since $\operatorname{coz}(f) = \bigcup_{n\in\mathbb{N}} A_n$, we deduce that $(A, \mathscr{A}_A)$ is a σ-compact measurable space. □

Proposition 4.5. Let $(X, \mathscr{A})$ be a T-measurable space and

$$C = \bigcup\{A \in \mathscr{A} : A \text{ is compact}\}.$$

Then, the following statements are true.

(1) $A \subseteq X \setminus C$ if and only if $f(A) = \{0\}$ for every $f \in M_\infty(X, \mathscr{A})$.

(2) $M_\infty(X, \mathscr{A}) \subseteq \bigcap\{M_x^* : x \in X \setminus C\}$, where $M_x^* = \{f \in M^*(X) : f(x) = 0\}$ for every $x \in X$.

Proof . (1). *Necessity.* By the definition of $M_\infty(X, \mathscr{A})$, $f(X \setminus C) = \{0\}$ for every $f \in M_\infty(X, \mathscr{A})$.

Sufficiency. Otherwise, there exist an element a in $A \cap C$ and a compact element B of $\mathscr{A}$ such that $a \in B$, showing that $\chi_B \in M_\infty(X, \mathscr{A})$ and $\chi_B(A) \neq \{0\}$, and this is a contradiction.

(2). By the first statement, this is evident. □

In the second statement of Proposition 4.5, the inclusion may be proper. This is the content of the following example.

Example 4.6. Consider $(X, \mathscr{A}) := (\mathbb{N}, P(\mathbb{N}))$. This is a locally compact T-measurable space, and the function $f : X \to \mathbb{R}$ defined by $x \mapsto \frac{x+1}{x}$ is not in $M_\infty(X, \mathscr{A})$. Therefore, $M_\infty(X, \mathscr{A}) \subsetneq M^*(X, \mathscr{A}) = \bigcap\{M_x^* : x \in X \setminus C\}$, since $X \setminus C = \emptyset$.

5 When is $M_\infty(X,\mathscr{A})$ a regular ring?

In this section, we observe that $M_\infty(X,\mathscr{A})$ is a regular ring if and only if it is an ideal of $M(X,\mathscr{A})$.

Example 5.1. Consider $(X,\mathscr{A}) := (\mathbb{N}, P(\mathbb{N}))$. This is a σ-compact T-measurable space, and the function $f:\mathbb{N}\to\mathbb{R}$ defined by $x\mapsto \dfrac{1}{x}$ is a unit element of $M(X,\mathscr{A})$ such that $f\in M_\infty(\mathbb{N})$. Since $M_\infty(X,\mathscr{A})\subseteq M^*(X,\mathscr{A})\subset M(X,\mathscr{A})$, we conclude that $M_\infty(X,\mathscr{A})$ is not an ideal of $M(X,\mathscr{A})$.

Definition 5.2. Let $(X,\mathscr{A})$ be a measurable space. A subset U of X is called *relatively pseudocompact* if $f(U)$ is a bounded subset of $\mathbb{R}$ for all $f\in M(X,\mathscr{A})$.

The next result is a new characterization of compact elements of a T-measurable space in terms of relatively pseudocompact measurable subspaces.

Proposition 5.3. Let $(X,\mathscr{A})$ be a T-measurable space and $A\in\mathscr{A}$. If $(A,\mathscr{A}_A)$ is a σ-compact measurable subspace of X, then the following statements are equivalent.

(1) A is a relatively pseudocompact measurable subspace of X.
(2) A is a compact element of $\mathscr{A}$.
(3) A is a finite subset of X.

Proof . (1) $\Rightarrow$ (2). If A is σ-compact, and not compact, then $(A,\mathscr{A}_A)\cong(\mathbb{N},\mathcal{P}(\mathbb{N}))$ by Propositions 3.2 and 3.6. Hence, the function $f:\mathbb{N}\to\mathbb{R}$ defined by $x\mapsto x$ is an element of $M(A)$. Since $A\in\mathscr{A}$, there exists an element $\bar f$ in $M(X,\mathscr{A})$ such that $\bar f|_A = f$. By statement (1), $f(A)=\bar f(A)$ is bounded, which is a contradiction.

(2) $\Rightarrow$ (1). By Proposition 3.2, A is a finite set. So, A is a relatively pseudocompact measurable subspace of X. □

Proposition 5.4. Let $(X,\mathscr{A})$ be a measurable space. Then, the ring $M_\infty(X,\mathscr{A})$ is an ideal of $M^*(X,\mathscr{A})$.

Proof . Consider $f\in M_\infty(X,\mathscr{A})$. Then $A:=\{x\in X: |f(x)|\geq 1\}$ is a compact element of $\mathscr{A}$, and hence $f[A]$ is a bounded subset of $\mathbb{R}$, from which it follows that $f\in M^*(X,\mathscr{A})$. Therefore, by Proposition 4.2, $M_\infty(X,\mathscr{A})$ is a subring of $M^*(X,\mathscr{A})$. Now, we assume that $f\in M_\infty(X,\mathscr{A})$ and $g\in M^*(X,\mathscr{A})$. Hence, there exists an element n_0 in $\mathbb{N}$ such that $|g(x)|\leq n_0$ for all $x\in X$. Since $\{x\in X: |f(x)|\geq \frac{1}{nn_0}\}$ is a compact element of $\mathscr{A}$ and

$$\left\{x\in X: |fg(x)|\geq \frac{1}{n}\right\}\subseteq\left\{x\in X: |f(x)|\geq\frac{1}{nn_0}\right\},$$

we conclude that $\{x \in X : |fg(x)| \geq \frac{1}{n}\}$ is a compact element of $\mathscr{A}$. Hence $fg \in M_\infty(X, \mathscr{A})$, and so $M_\infty(X, \mathscr{A})$ is an ideal of $M^*(X, \mathscr{A})$. □

Proposition 5.5. Let $(X, \mathscr{A})$ be a T-measurable space, and $A \in \mathscr{A}$ be a σ-compact element of $\mathscr{A}$. Then, A is a compact element of $\mathscr{A}$ if and only if $f|_A \in M^*(A, \mathscr{A}_A)$ for every $f \in M(X, \mathscr{A})$.

Proof . *Necessity.* If A is a compact element of $\mathscr{A}$, then by Proposition 3.7, $M(A, \mathscr{A}_A) = M^*(A, \mathscr{A}_A)$, from which it follows that $f|_A \in M^*(A, \mathscr{A}_A)$ for every $f \in M(X, \mathscr{A})$.

Sufficiency. Let A be an element of $\mathscr{A}$ which is not compact. Since, by Propositions 3.6 and 3.7, $(A, \mathscr{A}_A) \cong (\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and $M(A, \mathscr{A}_A) \neq M^*(A, \mathscr{A}_A)$, we conclude that there exists an element g in $M(A, \mathscr{A}_A) \setminus M^*(A, \mathscr{A}_A)$. Thus, if we define $f : X \to \mathbb{R}$ by $f(x) = g(x)$ when $x \in A$ and $f(x) = 0$ otherwise, then $f \in M(X, \mathscr{A})$ and $f|_A = g \notin M^*(A, \mathscr{A}_A)$, which is a contradiction. □

Definition 5.6. A measurable space $(X, \mathscr{A})$ is *locally compact* if there exists a compact element A of $\mathscr{A}$ such that $x \in A$ for every $x \in X$. Also, a non-empty subset Y of X is a *locally compact measurable subspace* of X provided that $(Y, \mathscr{A}_Y)$ is locally compact.

Remark 5.7. Every σ-compact measurable space is a locally compact measurable space.

Lemma 5.8. Let $(X, \mathscr{A})$ be a measurable space, and $\{f_n\}_{n\in\mathbb{N}} \subseteq M(X, \mathscr{A})$ be such that $0 \leq f_n(x) \leq f_{n+1}(x)$ for every $n \in \mathbb{N}$ and every $x \in X$. If the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f pointwise on X, then $f \in M(X, \mathscr{A})$.

Proof . Consider $r \in \mathbb{R}$. Since $0 \leq f_n(x) \leq f_{n+1}(x)$ for every $(x, n) \in X \times \mathbb{N}$, we conclude from $\lim_{n\to\infty} f_n(x) = f(x)$ that

$$\{x \in X : f(x) > r\} = \bigcup_{n\in\mathbb{N}} \{x \in X : f_n(x) > r\} \in \mathscr{A}.$$

Therefore, $f \in M(X, \mathscr{A})$. □

Proposition 5.9. Let $(X, \mathscr{A})$ be a T-measurable space. Then, $M_\infty(X, \mathscr{A})$ is an ideal of $M(X, \mathscr{A})$ if and only if every σ-compact element of $\mathscr{A}$ is relatively pseudocompact.

Proof . *Necessity.* Let $Y \in \mathscr{A}$ be a σ-compact element of $\mathscr{A}$ which is not a relatively pseudocompact element. Then, there exists an element g in $M(X, \mathscr{A})$ such that $g(Y)$ is not a bounded subset of $\mathbb{R}$. Hence, there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in Y such that $g(a_n) \geq 2^n$ and $g(a_{n+1}) > g(a_n) + 1$. Since Y is σ-compact, we deduce that there exists an element B_n

of $\mathscr{A}$ such that $a_n \in B_n \cap Y = A_n$, and A_n is a compact element of $\mathscr{A}_Y$ for every $n \in \mathbb{N}$. Since $Y \in \mathscr{A}$, we conclude that A_n is a compact element of $\mathscr{A}$ for every $n \in \mathbb{N}$. We let

$$U_n = g^{-1}\left(g(a_n) - \frac{1}{4}, g(a_n) + \frac{1}{4}\right) \cap A_n$$

for every $n \in \mathbb{N}$. Then U_i is a compact element of $\mathscr{A}$, $a_i \in U_i$ and $U_i \cap U_j = \emptyset$ for all $i, j \in \mathbb{N}$ with $i \neq j$. For $n \in \mathbb{N}$, let

$$s_n(x) = \sum_{i=1}^{n} \frac{\chi_{U_n}(x)}{2^n}$$

for every $x \in X$. By Lemma 5.8,

$$f(x) := \lim_{n\to\infty} s_n(x) = \sum_{i=1}^{\infty} \frac{\chi_{U_n}(x)}{2^n} \in M(X, \mathscr{A}).$$

For every fixed $m \in \mathbb{N}$, $K_m := \bigcup_{i=1}^{m} U_i$ is a compact element of $\mathscr{A}$. It is clear that $\chi_{U_n}(x) = 0$ for every $n \leq m$ and every $x \in K_m^c$. So,

$$f(x) := \Sigma_{n=1}^{\infty} \frac{\chi_{U_n}(x)}{2^n} \leq \Sigma_{n=m+1}^{\infty} \frac{1}{2^n} \leq \frac{1}{2^m} < \frac{1}{m}$$

for every $x \in K_m^c$. Since

$$\left\{x \in X : |f(x)| \geq \frac{1}{m}\right\} \subseteq K_m,$$

we conclude that $\{x \in X : |f(x)| \geq \frac{1}{m}\}$ is a compact element of $\mathscr{A}$. Hence, $f \in M_\infty(X, \mathscr{A})$. On the other hand, since

$$\{a_n : n \in \mathbb{N}\} \subseteq \left\{x \in X : |fg(x)| \geq \frac{1}{n}\right\},$$

we conclude from Proposition 3.2 that $\{x \in X : |fg(x)| \geq \frac{1}{n}\}$ is not a compact element of $\mathscr{A}$, which implies that $fg \notin M_\infty(X, \mathscr{A})$. Therefore, $M_\infty(X, \mathscr{A})$ is not an ideal of $M(X, \mathscr{A})$, which is a contradiction.

Sufficiency. Consider $f \in M_\infty(X, \mathscr{A})$ and $g \in M(X, \mathscr{A})$. Since, by Proposition 4.4, $\mathrm{coz}(f)$ is a σ-compact element of $\mathscr{A}$, we conclude from our hypothesis that $g(\mathrm{coz}(f))$ is a bounded subset of $\mathbb{R}$, from which we find that there exists an element k in $\mathbb{N}$ such that $|g(x)| \leq k$ for every $x \in \mathrm{coz}(f)$. For every fixed $n \in \mathbb{N}$,

$$\left\{x \in X : |fg(x)| \geq \frac{1}{n}\right\} = \left\{x \in \mathrm{coz}(f) : |f(x)g(x)| \geq \frac{1}{n}\right\} \subseteq \left\{x \in \mathrm{coz}(f) : |f(x)| \geq \frac{1}{nk}\right\}.$$

Since $\{x \in \mathrm{coz}(f) : |f(x)| \geq \frac{1}{nk}\}$ is a compact element of $\mathscr{A}$, we conclude that $\{x \in X : |fg(x)| \geq \frac{1}{n}\}$ is a compact element of $\mathscr{A}$. Hence, $fg \in M_\infty(X, \mathscr{A})$. So, by Proposition 4.2, $M_\infty(X, \mathscr{A})$ is an ideal of $M(X, \mathscr{A})$. $\square$

Definition 5.10. Let $(X, \mathscr{A})$ be a measurable space, and $M_K(X, \mathscr{A})$ denote the family of all functions $f \in M(X, \mathscr{A})$ for which $coz(f)$ is a compact element of $\mathscr{A}$.

Remark 5.11. Let $(X, \mathscr{A})$ be a T-measurable space. Consider $f \in M_K(X, \mathscr{A})$ and $n \in \mathbb{N}$. Since $coz(f)$ is a compact element of $\mathscr{A}$ and

$$\left\{x \in X : |f(x)| \geq \frac{1}{n}\right\} \subseteq \operatorname{coz}(f),$$

we conclude that $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is a compact element of $\mathscr{A}$. Therefore,

$$M_K(X, \mathscr{A}) \subseteq M_\infty(X, \mathscr{A}).$$

Also, by Proposition 3.2,

$$M_K(X, \mathscr{A}) = \{f \in M(X, \mathscr{A}) : \ \operatorname{coz}(f) \text{ is finite}\}.$$

The ring A (commutative, with 1) is regular if for each element a, there exists an element b such that $a^2 b = a$.

Proposition 5.12. If $(X, \mathscr{A})$ is a T-measurable space, then $M_K(X, \mathscr{A})$ is a regular ring.

Proof . Consider $f \in M_K(X, \mathscr{A})$. We define $g : X \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } x \in Z(f) \\ \frac{1}{f(x)} & \text{if } x \notin Z(f), \end{cases}$$

for every $x \in X$. Then, $g \in M_K(X, \mathscr{A})$ and $f = gf^2$. Therefore, $M_K(X, \mathscr{A})$ is a regular ring. □

The following example shows that $M_\infty(X, \mathscr{A})$ may not be regular.

Example 5.13. Consider $(X, \mathscr{A}) := (\mathbb{N}, \mathcal{P}(\mathbb{N}))$. We define $f : X \longrightarrow \mathbb{R}$ by $x \mapsto \frac{1}{x}$. Then, $f \in M_\infty(X, \mathscr{A})$ and $\operatorname{coz}(f) = X$. If $g \in M(X, \mathscr{A})$ such that $f = gf^2$, then g is the inclusion function and $g \notin M_\infty(X, \mathscr{A})$. Therefore, $M_\infty(X, \mathscr{A})$ is not a regular ring.

Given a T-measurable space $(X, \mathscr{A})$, the following result presents some conditions equivalent to the regularity of $M_\infty(X, \mathscr{A})$ as a ring.

Proposition 5.14. If $(X, \mathscr{A})$ is a T-measurable space, then the following statements are equivalent.

(1) $M_\infty(X,\mathscr{A})$ is an ideal of $M(X,\mathscr{A})$.

(2) Every σ-compact element of $\mathscr{A}$ is compact.

(3) Every σ-complement of $\mathscr{A}$ is a finite subset of X.

(4) If $\{A_n\}_{n\in\mathbb{N}}$ is a family of compact elements of $\mathscr{A}$ such that

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq A_{n+1} \subseteq \cdots ,$$

then there exists an element k in $\mathbb{N}$ such that $A_k = A_{k+i}$ for all $i \in \mathbb{N}$. In other words, every ascending chain of compact elements of $\mathscr{A}$, ordered by the relation $\subseteq$, is finite.

(5) $M_\infty(X,\mathscr{A}) = M_K(X,\mathscr{A})$.

(6) $M_\infty(X,\mathscr{A})$ is a regular ring.

Proof . (1) $\Rightarrow$ (2). If $A \in \mathscr{A}$ is σ-compact and not compact, then by Proposition 5.3, A is not a relatively pseudocompact measurable subspace of X, and so by Proposition 5.9, $M_\infty(X,\mathscr{A})$ is not an ideal of $M(X,\mathscr{A})$, which is a contradiction.

(2) $\Rightarrow$ (3). By Proposition 3.2, this is evident.

(3) $\Rightarrow$ (4). Let $\{A_n\}_{n\in\mathbb{N}}$ be a family of compact elements of $\mathscr{A}$ such that

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq A_{n+1} \subseteq \cdots .$$

Then, $\bigcup_{n\in\mathbb{N}} A_n$ is a family of σ-compact elements of $\mathscr{A}$. By statement (3), $\bigcup_{n\in\mathbb{N}} A_n$ is a family of compact elements of $\mathscr{A}$. We deduce from Proposition 3.2 that $\bigcup_{n\in\mathbb{N}} A_n$ is a finite set, and this implies that there exists an element k in $\mathbb{N}$ such that $A_k = A_{k+i}$ for all $i \in \mathbb{N}$.

(4) $\Rightarrow$ (1). Let $A \in \mathscr{A}$ be a σ-compact element of $\mathscr{A}$. Then, by Remark 3.5, there exists an ascending chain $\{A_n : n \in \mathbb{N}\}$ of compact elements of $\mathscr{A}$ such that $A = \bigcup_{n\in\mathbb{N}} A_n$. Hence, by our hypothesis, there exists an element n in $\mathbb{N}$ such that $A = A_n$, from which it follows that A is compact. Then, A is a finite set by Proposition 3.2, and so it is relatively pseudocompact. Therefore, by Proposition 5.9, $M_\infty(X,\mathscr{A})$ is an ideal of $M(X,\mathscr{A})$.

(4) $\Rightarrow$ (5). Consider $f \in M_\infty(X,\mathscr{A})$. For every $n \in \mathbb{N}$, let

$$A_n = \left\{x \in X : |f(x)| \geq \frac{1}{n}\right\},$$

so that

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq A_{n+1} \subseteq \cdots .$$

Hence, by our hypothesis, there exists an element n in $\mathbb{N}$ such that

$$A = \bigcup_{n\in\mathbb{N}} \left\{x \in X : |f(x)| \geq \frac{1}{n}\right\} = \operatorname{coz}(f)$$

is a compact element of $\mathscr{A}$, showing that $f \in M_K(X, \mathscr{A})$. We conclude from Remark 5.11 that $M_\infty(X, \mathscr{A}) = M_K(X, \mathscr{A})$.

(5) ⇒ (6). By Proposition 5.12, $M_\infty(X, \mathscr{A})$ is regular.

(6) ⇒ (2). If $A \in \mathscr{A}$ is σ-compact, then by Proposition 4.4 there exists an element f in $M_\infty(X, \mathscr{A})$ such that $\text{coz}(f) = A$. By the hypothesis, there exists an element g in $M_\infty(X, \mathscr{A})$ such that $gf^2 = f$, which implies $Z(f) = \{x \in X : fg(x) \neq 1\}$. We find that $g(x) = \frac{1}{f(x)}$ for every $x \in \text{coz}(f)$. Since $f \in M_\infty(X, \mathscr{A}) \subseteq \mathscr{A}^*(X)$, there exists an element k in $\mathbb{N}$ such that $|f(x)| \leq k$, for every $x \in X$. Hence, $|g(x)| \geq \frac{1}{k}$ for any $x \in \text{coz}(f)$, that is, $\text{coz}(f) \subseteq \{x \in X : |g(x)| \geq \frac{1}{k}\}$. Since $g \in M_\infty(X, \mathscr{A})$, we conclude that $\{x \in X : |g(x)| \geq \frac{1}{k}\}$ is compact. Thus, $A = \text{coz}(f)$ is compact. □

Proposition 5.15. Let $(X, \mathscr{A})$ be a T-measurable space. Then, $M_\infty(X, \mathscr{A}) \cong \mathbb{R}$ if and only if $\mathscr{A}$ has a unique non-empty compact element.

Proof . *Necessity.* Let $A, B \in \mathscr{A} \setminus \{\emptyset\}$ be distinct compact elements of $\mathscr{A}$. Since $A \cup B$ is a compact element of $\mathscr{A}$ and $|A \cup B| \geq 2$, we conclude from Corollary 3.3 that $\mathbb{R}^2$ is a subring of $\mathscr{A}(A \cup B) = \mathscr{A}^*(A \cup B) = M_\infty(A \cup B)$, which is a subring of $M_\infty(X, \mathscr{A}) \cong \mathbb{R}$, and this is a contradiction.

Sufficiency. Let A be the unique non-empty compact element of $\mathscr{A}$. By Proposition 3.2, there exists an element a of X such that $A = \{a\}$. If $0 \neq f \in M_\infty(X, \mathscr{A})$, then by Proposition 4.4, $\text{coz}(f)$ is σ-compact, and so $\text{coz}(f) = \{a\}$. Hence, $M_\infty(X, \mathscr{A}) =< \chi_A >$, from which it follows that $M_\infty(X, \mathscr{A}) \cong \mathbb{R}$. □ In the next proposition, we characterize compact T-measurable spaces in terms of the ring of all real measurable functions.

Proposition 5.16. Let $(X, \mathscr{A})$ be a T-measurable space. Then, $M_\infty(X, \mathscr{A}) = M(X, \mathscr{A})$ if and only if $(X, \mathscr{A})$ is a compact measurable space.

Proof . *Necessity.* Since $\mathbf{1} \in M(X, \mathscr{A}) = M_\infty(X, \mathscr{A})$, we conclude that $X = \{x \in X : |\mathbf{1}(x)| \geq 1\}$ is a compact measurable space.

Sufficiency. By Proposition 3.2, $M(X, \mathscr{A}) = M^*(X, \mathscr{A})$. Since $\mathbf{1} \in M(X, \mathscr{A})$, we conclude from Proposition 5.4 that $M_\infty(X, \mathscr{A}) = M(X, \mathscr{A})$. □

6 On locally compact measurable spaces

In this section, we show that when an algebraic or lattice property holds for $M_\infty(X, \mathscr{A})$ with $(X, \mathscr{A})$ being a locally compact measurable space, it also holds for $M_\infty(X, \mathscr{A})$ with $(X, \mathscr{A})$ being arbitrary.

Example 6.1. Let X be a non-empty set, and let $\mathscr{A}$ be the collection of all sets $E \subseteq X$ such that either E or $X \setminus E$ is at most countable. Recall from [20] that $(X, \mathscr{A})$ is a measurable space. It is clear that $(X, \mathscr{A})$ is a locally compact T-measurable space. Also, if X is an uncountable set, then $\mathscr{A} \neq \mathcal{P}(X)$.

For any subset A of $M(X, \mathscr{A})$, we write $\mathrm{Coz}_{\mathscr{A}}[A] = \{\mathrm{coz}(f) : f \in A\}$.

Lemma 6.2. Let $(X, \mathscr{A})$ be a T-measurable space, and

$$\mathcal{B} := \{f^{-1}(O) : f \in M_\infty(X, \mathscr{A}),\, O \in \mathfrak{O}(\mathbb{R})\}.$$

Then, $< \mathrm{Coz}(M_K(X, \mathscr{A})) >=< \mathrm{Coz}(M_\infty(X, \mathscr{A})) >=< \mathcal{B} >$.

Proof . Suppose that $f \in M_\infty(X, \mathscr{A})$. Since

$$f^{-1}\left(\frac{-1}{n}, \frac{1}{n}\right) = X \setminus \left\{x \in X : |f(x)| \geq \frac{1}{n}\right\} \in < \mathrm{Coz}(M_K(X, \mathscr{A})) >$$

for every $n \in \mathbb{N}$, we conclude that $Z(f) \in < \mathrm{Coz}(M_K(X, \mathscr{A})) >$. If $0 \notin O \in \mathfrak{O}(\mathbb{R})$ and $f \in M_\infty(X, \mathscr{A})$, then

$$f^{-1}(O) = \mathrm{coz}(\chi_{f^{-1}(O)}) \in < \mathrm{Coz}(M_K(X, \mathscr{A})) >,$$

from which it follows that if $0 \in O \in \mathfrak{O}(\mathbb{R})$ and $f \in M_\infty(X, \mathscr{A})$, then

$$f^{-1}(O) = Z(f) \cup f^{-1}(O \setminus \{0\}) \in < \mathrm{Coz}(M_K(X, \mathscr{A})) >,$$

and we conclude that

$$\mathcal{B} \subseteq < \mathrm{Coz}(M_K(X, \mathscr{A})) > \subseteq < \mathrm{Coz}(M_\infty(X, \mathscr{A})) > .$$

Now, we deduce from

$$\mathrm{Coz}(M_K(X, \mathscr{A})) \subseteq \mathrm{Coz}(M_\infty(X, \mathscr{A})) \subseteq \mathcal{B}$$

that $< \mathrm{Coz}(M_K(X, \mathscr{A})) >=< \mathrm{Coz}(M_\infty(X, \mathscr{A})) >=< \mathcal{B} >$. $\square$

Proposition 6.3. Let $(X, \mathscr{A})$ be a T-measurable space and $\mathcal{A} = \{\{x\} : x \in X\}$. Then, the following statements are equivalent.

(1) The measurable space $(X, \mathscr{A})$ is a locally compact T-measurable space, and if $(X, \mathscr{A}')$ is a locally compact T-measurable space, then $\mathscr{A} \subseteq \mathscr{A}'$. ($\mathscr{A}$ is the smallest locally compact T-measurable space on X.)

(2) The σ-algebra $\mathscr{A}$ is generated by $\mathcal{A}$.

(3) The σ-algebra $\mathscr{A}$ is the collection of all sets $E \subseteq X$ such that either E or $X \setminus E$ is at most countable.
(4) The σ-algebra $\mathscr{A}$ is generated by $\mathrm{coz}(M_\infty(X, \mathscr{A}))$.
(5) The σ-algebra $\mathscr{A}$ is generated by $\mathrm{coz}(M_K(X, \mathscr{A}))$.
(6) The σ-algebra $\mathscr{A}$ is the same weak measure induced by $M_\infty(X, \mathscr{A})$.

Proof . (1) $\Rightarrow$ (2). It is clear that $(X, < \mathcal{A} >)$ is a locally compact T-measurable space. By the first statement, $\mathscr{A} \subseteq < \mathcal{A} >$. We find that $\mathcal{A} \subseteq \mathscr{A}$, for every locally compact T-measurable space $(X, \mathscr{A})$. We deduce that $< \mathcal{A} >= \mathscr{A}$ is the smallest locally compact T-measurable space on X.

(2) $\Rightarrow$ (1). This is evident.

(2) $\Leftrightarrow$ (3). Let $\mathscr{A}'$ be the collection of all sets $E \subseteq X$ such that either E or $X \setminus E$ is at most countable. By Example 6.1, $(X, \mathscr{A}')$ is a locally compact T-measurable space. Therefore, $\mathscr{A}' \subseteq < \mathcal{A} >$. Also, $\mathcal{A} \subseteq \mathscr{A}'$ and hence, $< \mathcal{A} > \subseteq \mathscr{A}'$. We conclude that $< \mathcal{A} >= \mathscr{A}'$.

(2) $\Rightarrow$ (4). Since for every $x \in X$, $\{x\} = \mathrm{coz}(\chi_{\{x\}}) \in \mathrm{coz}(M_\infty(X, \mathscr{A}))$, we find that $\mathcal{A} \subseteq \mathrm{coz}(M_\infty(X, \mathscr{A}))$. Consider $f \in M_\infty(X, \mathscr{A})$. Since $\mathrm{coz}(f) = \bigcup_{n \in \mathbb{N}} \{x \in X : |f(x)| \geq \frac{1}{n}\}$ is a σ-compact element of $\mathscr{A}$, we conclude from Proposition 3.6 that $\mathrm{coz}(f)$ is at most countable, from which it follows that $\mathrm{coz}(f) \in < \mathcal{A} >$. Therefore, $< \mathcal{A} >=< \mathrm{Coz}(M_\infty(X, \mathscr{A})) >$.

(4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6). By Lemma 6.2, this is evident.

(4) $\Rightarrow$ (2). Consider $\{f_n\}_{n \in \mathbb{N}} \subseteq M_\infty(X, \mathscr{A})$. Then $\bigcup_{n \in \mathbb{N}} \mathrm{coz}(f_n)$ is σ-compact, because $\mathrm{coz}(f_n)$ is σ-compact for any $n \in \mathbb{N}$. By Proposition 4.4, there exists an element f in $M_\infty(X, \mathscr{A})$ such that $\mathrm{coz}(f) = \bigcup_{n \in \mathbb{N}} \mathrm{coz}(f_n)$. Then, $\{\mathrm{coz}(f) : f \in M_\infty(X, \mathscr{A})\}$ is closed under countable unions. Also, since $\bigcap_{n \in \mathbb{N}} \mathrm{coz}(f_n)$ is σ-compact, we deduce from Proposition 4.4 that $\{\mathrm{coz}(f) : f \in M_\infty(X)\}$ is closed under countable intersections. Therefore, by Proposition 4.4,

$$\begin{aligned} < \mathrm{Coz}(M_\infty(X, \mathscr{A})) > &= \{\mathrm{coz}(f) : f \in M_\infty(X, \mathscr{A})\} \cup \{X \setminus \ \mathrm{coz}(f) : f \in M_\infty(X, \mathscr{A})\} \\ &= \{A \subseteq X : \ A \text{ or } X \setminus A \text{ is at most countable}\}. \end{aligned}$$

□

Proposition 6.4. Let $(X, \mathscr{A})$ be a T-measurable space. Then, the following statements are equivalent.

(1) X is a locally compact space.
(2) For any $x_0 \in X$ and $F \in \mathscr{A}$ with $x_0 \notin F$, there exists an element f in $M_\infty(X, \mathscr{A})$ such that $f(x_0) = 1$ and $f([F]) = \{0\}$.
(3) $M_\infty(X, \mathscr{A})$ is a free ideal of $M^*(X, \mathscr{A})$.

(4) $M_K(X,\mathscr{A})$ is a free ideal of $M^*(X,\mathscr{A})$ and $M(X,\mathscr{A})$.

Proof . (1) ⇒ (2). Let $a \in X$ and $F \in \mathscr{A}$ with $a \notin F$. By the first statement, there exists a compact element A of $\mathscr{A}$ such that $a \in A$, and we conclude that $\chi_{A\setminus F} \in M_K(X,\mathscr{A}) \subseteq M_\infty(X,\mathscr{A})$.

(2) ⇒ (3). If $|X| < \aleph_0$, then X is compact by Proposition 3.2, and we conclude from Proposition 5.16 that $\mathbf{1} \in M(X,\mathscr{A}) = M_\infty(X,\mathscr{A}) = M_K(X,\mathscr{A})$. Hence, $M_\infty(X,\mathscr{A})$ is free. Now, we can assume that $|X| \geq \aleph_0$. Consider $a \in X$. Then, by the hypothesis, there exists an element B in $\mathscr{A}$ such that $a \notin B$ and by the second statement, there exists an element f in $M_\infty(X,\mathscr{A})$ such that $a \in \operatorname{coz}(f)$. Hence $\bigcup_{f\in\mathscr{A}_\infty(X)} \operatorname{coz}(f) = X$, from which it follows that $M_\infty(X,\mathscr{A})$ is free.

(3) ⇒ (1). Consider $a \in X$. Since $M_\infty(X,\mathscr{A})$ is free, we find that there exists an element f in $M_\infty(X,\mathscr{A})$ such that $a \in \operatorname{coz}(f) = \bigcup_{n\in\mathbb{N}} |f|^{-1}(\frac{1}{n},\infty)$, from which it follows that there exists an element n in $\mathbb{N}$ such that $a \in |f|^{-1}(\frac{1}{n},\infty)$ and $|f|^{-1}(\frac{1}{n},\infty)$ is compact.

(1) ⇒ (4). Let $x \in X$ be given. Then $\{x\} \in \mathscr{A}$, and hence $\chi_{\{x\}} \in M_K(X,\mathscr{A})$. Therefore, $X \subseteq Coz(M_K(X,\mathscr{A}))$.

(4) ⇒ (2). If $x \in X$, then $x \in \bigcup Coz(M_K(X,\mathscr{A}))$. There exists f in $M_K(X,\mathscr{A})$ such that $x \in coz(f)$, which is compact. Therefore, X is locally compact. □

Lemma 6.5. Let $(X,\mathscr{A})$ be a T-measurable space and

$$Y = \{x \in X : \{x\} \in \mathscr{A}\}.$$

Then, the following statements are true.

(1) $(Y,\mathscr{A}_Y)$ is a locally compact T-measurable space.
(2) For every subset A of X, $A \in \mathscr{A}$ is compact if and only if $A \in \mathscr{A}_Y$ is compact.
(3) For every subset A of X, $A \in \mathscr{A}$ is σ-compact if and only if $A \in \mathscr{A}_Y$ is σ-compact.
(4) For any $f \in M_\infty(X,\mathscr{A})$, $\operatorname{coz}(f) \subseteq Y$.
(5) If $M_\infty(X,\mathscr{A}) \neq \{0\}$, then $Y \neq \emptyset$.

Proof . (5). Suppose that $\mathbf{0} \neq f \in M_\infty(X,\mathscr{A})$. Then, there exists an element n in $\mathbb{N}$ such that $\emptyset \neq \{x \in X : |f(x)| \geq \frac{1}{n}\}$, and it is compact. Hence, $\emptyset \neq \{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq Y$.

The rest of the proof can be easily completed. □

Next, we show that in the study of rings of real measurable functions vanishing at infinity on a measurable space and rings of real measurable functions with compact support on a measurable space, there is no need to deal with measurable spaces that are not locally compact T-measurable spaces.

Proposition 6.6. If $(X, \mathscr{A})$ is a T-measurable space and $M_\infty(X, \mathscr{A}) \neq \{0\}$, then there exists a locally compact measurable space $(Y, \mathscr{A}')$ such that

(1) $M_\infty(X, \mathscr{A}) \cong M_\infty(Y, \mathscr{A}')$, and
(2) $M_K(X, \mathscr{A}) \cong M_K(Y, \mathscr{A}')$.

Proof . (1). Consider $Y = \{x \in X : \{x\} \in \mathscr{A}\}$ and $(Y, \mathscr{A}') := (Y, \mathscr{A}_Y)$. We define $\theta : M_\infty(X, \mathscr{A}) \longrightarrow M_\infty(Y, \mathscr{A}')$ by $f \mapsto f|_Y$ for any $f \in M_\infty(X, \mathscr{A})$. Consider $f \in M_\infty(X, \mathscr{A})$ and $n \in \mathbb{N}$. Since ${f|_Y}^{-1}(V) = f^{-1}(V) \cap Y \in \mathscr{A}'$ for any $V \in \mathfrak{O}(\mathbb{R})$, it follows that $f|_Y \in M(Y)$. Using

$$\{x \in X : x \in A \text{ for some compact element } A \text{ of } \mathscr{A}\} = \{x \in X : \{x\} \in \mathscr{A}\}$$

and $\operatorname{coz}(f) \subseteq Y$ we obtain

$$\left\{y \in Y : |f|_Y|(y) \geq \frac{1}{n}\right\} = \left\{x \in X : |f|(x) \geq \frac{1}{n}\right\}.$$

Since $\{x \in X : |f|(x) \geq \frac{1}{n}\}$ is compact in $(X, \mathscr{A})$, we deduce from Lemma 6.5 that $\{y \in Y : |f|_Y|(y) \geq \frac{1}{n}\}$ is compact in $(Y, \mathscr{A}')$, from which it follows that $f|_Y \in M_\infty(Y, \mathscr{A}')$. If $\diamond \in \{+, ., \vee, \wedge\}$, then $\theta(f \diamond g) = (f \diamond g)|_Y = f|_Y \diamond g|_Y$, and so θ is an f-ring homomorphism. If $f \in \ker\theta$, then $f|_Y = \mathbf{0}$. Since $f \in M_\infty(X, \mathscr{A})$, we infer from Lemma 6.5 that $\operatorname{coz}(f) \subseteq Y$. Thus, $f = \mathbf{0}$. Consider $g \in M_\infty(Y, \mathscr{A}')$. Since $\operatorname{coz}(g)$ is a σ-compact element of $\mathscr{A}'$, we conclude from Lemma 6.5 that $\operatorname{coz}(g)$ is a σ-compact element of $\mathscr{A}$. Therefore, $X \setminus \operatorname{coz}(g) \in \mathscr{A}$. Define

$$g^*(x) = \begin{cases} g(x) & \text{if } x \in \operatorname{coz}(g) \\ 0 & \text{if } x \in X \setminus \operatorname{coz}(g). \end{cases}$$

We show that $g^* \in M(X, \mathscr{A})$. For any $r \in \mathbb{R}$,

$$\{x \in X : g^*(x) > r\} = \{x \in Y : g(x) > r\}$$

if $r \geq 0$, and

$$\{x \in X : g^*(x) > r\} = \{x \in Y : g(x) \in (r, 0) \cup (0, \infty)\} \cup \big(X \setminus \operatorname{coz}(g)\big)$$

if $r < 0$. Then $\{x \in X : g^*(x) > r\} \in \mathscr{A}$, and hence $g^* \in M(X, \mathscr{A})$. Since

$$\left\{x \in X : |g^*|(x) > \frac{1}{n}\right\} = \left\{x \in Y : |g^*|(x) > \frac{1}{n}\right\}$$

is a compact element of $\mathscr{A}'$, we conclude from Lemma 6.5 that it is a compact element of $\mathscr{A}$. Therefore, $g^* \in M_\infty(X, \mathscr{A})$ and $\theta(g^*) = g^*|_Y = g$.

(2). We consider $\theta_K =: \theta|_{M_K(X,\mathscr{A})}$. By the first statement, θ_K is a monomorphism and $\operatorname{coz}(\theta_K(f)) = \operatorname{coz}(f)$. We deduce from Lemma 6.5 that $\theta_K(f) \in M_K(Y,\mathscr{A}')$. Let $g \in M_K(Y,\mathscr{A}')$ be given. Then, by the same method as the one used in the proof of the first statement, we obtain $g^* \in M_K(X,\mathscr{A})$ and $\theta_K(g^*) = g$. Therefore, $M_K(X,\mathscr{A}) \cong M_K(Y,\mathscr{A}')$. $\square$

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