



c_L -regular-open and c_L -regular-closed elements by clouser function in pointfree topology

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Abstract

In this paper, we consider the pairs (L, cl_L) , where L is a locale and $cl_L : L \rightarrow L$ is a closure function on L and also describe connections between closure functions and interior functions in a locale L . Moreover, we introduce c_L -regular-open and c_L -regular-closed elements in a closure function and study their fundamental properties. We show that the set of all c_L -regular-open elements and c_L -regular-closed elements in a locale L are frames with binary relation \leq_L .

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1 Introduction and preliminaries

Let $P(X)$ be the power set of the non-empty set X and $\text{cl}: P(X) \rightarrow P(X)$ be an arbitrary set-valued function such that

- (K0) $\text{cl}(\emptyset) = \emptyset$,
- (K1) $A \subseteq B$ implies $\text{cl}(A) \subseteq \text{cl}(B)$,
- (K2) $A \subseteq \text{cl}(A)$,
- (K3) $\text{cl}(A \cup B) \subseteq \text{cl}(A) \cup \text{cl}(B)$,
- (K4) $\text{cl}(\text{cl}(A)) = \text{cl}(A)$,

where $A, B \in P(X)$. This function is called Kuratowski closure operator which constructs a unique topology on X such that $F \subseteq X$ is closed, if $\text{cl}(F) = F$. This space is called a closure space and denoted by (X, cl) . Closure spaces are studied in [2], [8], [9] and [10]. An isotonic space is a pair (X, cl) such that $\text{cl}: P(X) \rightarrow P(X)$ satisfies the axioms (K0) and (K1). Recently, Habil and Elzenati [[5] and [6]], Stadler et al. in [12] and [13] and Boonpok in [1] studied the isotonic spaces and determined some basic principles of these spaces. In [4] and [3], Ersoy and Erol introduced bi-isotonic spaces. Haghdadi and Estaji in [7], introduced isotonic closure functions on a locale. They consider a pair $(L, \underline{\text{cl}}_L)$, where L is a locale and $\underline{\text{cl}}_L: \mathcal{S}(L) \rightarrow \mathcal{S}(L)$ is an isotonic function on the sublocales of L . Moreover, they introduced the concept of generalized $\underline{\text{cl}}_L$ - closed sublocales in isotonic closure locales and discuss some of their properties.

This paper is organized as follows. In Section 1, we review some basic notions together with some known properties of locales and isotonic spaces. In Section 2, we introduce and study isotonic closure functions. These are pairs of the form (L, cl_L) , where L is a locale and $\text{cl}_L: L \rightarrow L$ is an isotonic closure function on L . We describe connections between closure functions and interior functions in a locale L . In Section 3, we introduce c_L -regular-open and c_L -regular-closed elements in an isotonic closure function and study their fundamental properties. We show that the set of all c_L -regular-open elements and the set of all c_L -regular-closed elements in locale L are frames with binary relation \leq_L .

Pointfree topology focuses on the open sets rather than the points of a space, and deals with abstractly defined the lattice of open sets, called frames, and their homomorphisms. Formally, a *locale* L is defined as a special complete lattice (where the *top* and the *bottom* elements are denoted by 1 and 0, respectively) usually called a *frame*, in which finite meets distribute over arbitrary joins, that is, $a \wedge \bigvee S = \bigvee \{a \wedge b: b \in S\}$ for all $a \in L$ and $S \subseteq L$. A *sublocale* of a locale L is a subset S of L which is closed under arbitrary meets and $x \rightarrow s \in S$ for every $(x, s) \in L \times S$ (see [11]).

Definition 1.1. [11] Let L be a locale. A pseudocomplement of an element a denoted by a^* , is the largest element b such that $b \wedge a = 0$, if it exists. Thus, a^* is characterized by

the formula $a^* := \bigvee\{x: x \wedge a = 0\}$. Also the relative pseudodifference of a with respect to x denoted by $x \setminus a$ given by the formula $x \setminus a = \bigwedge\{y: x \leq a \vee y\}$. The (absolute) pseudodifference of an element a is the element $a^\# = 1 \setminus a := \bigwedge\{y: y \vee a = 1\}$.

In the following proposition we give some properties of pseudodifference and pseudocomplement.

Proposition 1.2. Let L be a locale and $a, b, x \in L$. Then

- (1) $a \vee a^\# = 1$ and $a \wedge a^* = 0$.
- (2) If $a \leq b$, then $a^\# \geq b^\#$ and $a^* \geq b^*$.
- (3) $a^{\#\#} \leq a$, and $a^{\#\#\#} = a^\#$, $a^{**} \geq a$ and $a^{***} = a^*$.
- (4) $0^\# = 1 = 0^*$, and $1^\# = 0 = 1^*$.
- (5) For every $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq L$, and $(\bigvee_{\lambda \in \Lambda} a_\lambda)^* = \bigwedge_{\lambda \in \Lambda} a_\lambda^*$.
- (6) If $a \wedge x = b \wedge x$, then $a \vee x^\# = b \vee x^\#$.

Proposition 1.3. Let L be a locale. Then, the following conditions are equivalent.

1. L is a Boolean algebra.
2. $x^{**} = x$ for all $x \in L$.
3. L is pseudocomplemented and $x = x^{\#\#}$ for all $x \in L$.
4. L is pseudocomplemented and $x^* = x^\#$ for all $x \in L$.

Definition 1.4. A nucleus in a locale L is a mapping $\nu: L \rightarrow L$ such that

- (N1) $a \leq \nu(a)$,
- (N2) If $a \leq b$ then $\nu(a) \leq \nu(b)$,
- (N3) $\nu\nu(a) = \nu(a)$, and
- (N4) $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$.

A map $f: L \rightarrow M$ (where L, M are locales) is said to be a *localic map* whenever for every $a \in L, b \in M$ and $S \subseteq L$,

- (L1) $f(\bigwedge S) = \bigwedge f[S]$ (and, in particular, $f(1) = 1$),
- (L2) $f(f^*(b) \rightarrow a) = b \rightarrow f(a)$, and
- (L3) $f(a) = 1 \Rightarrow a = 1$,

where $f^*: M \rightarrow L$ denotes the left adjoint of f provided by (L1). For a sublocale $S \subseteq L$, set $\nu_S(a) = \bigwedge\{s \in S: a \leq s\}$.

2 Isotonic closure functions on a locale

In this section, we define isotonic closure function $c_L: L \rightarrow L$ on a locale L and by this definition we define *c-closed* and *c-open* elements. Moreover, we define the neighborhood function $\mathcal{N}: L \rightarrow \mathcal{P}(L)$ and give relationship between closures of elements and neighborhood of elements. Let L be a locale and $c_L: L \rightarrow L$ be a function. Consider the following axioms for arbitrary $x, y \in L$,

- (K0) $c_L(\perp) = \perp$.
- (K1) $x \leq y$ implies $c_L(x) \leq c_L(y)$.
- (K2) $x \leq c_L(x)$.
- (K3) $c_L(x \vee y) \leq c_L(x) \vee c_L(y)$.
- (K4) $c_L(c_L(x)) = c_L(x)$.

Now, we define $\text{int}_L: L \rightarrow L$ by $\text{int}_L(x) = (c_L(x^\#))^\#$.

Proposition 2.1. Let L be a locale. The following statements hold.

1. If c_L satisfies in (K0), then $\text{int}_L(\top) = \top$.
2. If c_L satisfies in (K1), then int_L is an increasing function.
3. If c_L satisfies in (K2), then int_L is a contracting function.
4. If c_L satisfies in (K3), then $(\text{int}_L(x) \wedge \text{int}_L(y))^{\#\#} \leq \text{int}_L(x \wedge y)$ for every $x, y \in L$.
5. If $c_L: L \rightarrow L$ satisfies in (K1), (K2) and (K4), then int_L is an idempotent function.

Proof . (1) By condition (K0), $\text{int}_L(\top) = (c_L(\top^\#))^\# = (c_L(\perp))^\# = (\perp)^\# = \top$.

(2) Let $x, y \in L$ where $x \leq y$. By condition (K1), $c_L(y^\#) \leq c_L(x^\#)$ and so, $(c_L(x^\#))^\# \leq (c_L(y^\#))^\#$. Therefore, $\text{int}_L(x) \subseteq \text{int}_L(y)$ and so int_L is an increasing function.

(3) Let x be in L . By (K2), $x^\# \leq c_L(x^\#)$ and so, $(c_L(x^\#))^\# \leq x^{\#\#} \leq x$, which means that $\text{int}_L(x) \leq x$.

(4) Let $x, y \in L$ be given, then

$$\begin{aligned}
 \text{int}_L(x \wedge y) &= (c_L((x \wedge y)^\#))^\# \\
 &= (c_L(x^\# \vee y^\#))^\# \\
 &\geq (c_L(x^\#) \vee c_L(y^\#))^\# \\
 &= (c_L(x^\#) \vee c_L(y^\#))^{\#\#\#} \\
 &= ((c_L(x^\#))^{\#\#} \vee (c_L(y^\#))^{\#\#})^\# \\
 &= (\text{int}_L(x)^\# \vee \text{int}_L(y)^\#)^\# \\
 &= (\text{int}_L(x) \wedge \text{int}_L(y))^{\#\#}.
 \end{aligned}$$

(5) By (3), $\text{int}_L(\text{int}_L(x)) \leq \text{int}_L(x)$. To complete the proof, note that by (K1) and (K4) we can write

$$\begin{aligned} (c_L(x^\#))^{\#\#} \leq c_L(x^\#) &\Rightarrow c_L\left((c_L(x^\#))^{\#\#}\right) \leq c_L(c_L(x^\#)) \\ &\Rightarrow c_L\left((c_L(x^\#))^{\#\#}\right) \leq c_L(x^\#) \\ &\Rightarrow (c_L(x^\#))^\# \leq \left(c_L\left((c_L(x^\#))^{\#\#}\right)\right)^\# \\ &\Rightarrow \text{int}_L(x) \leq \left(c_L\left((\text{int}_L(x))^\#\right)\right)^\# \\ &\Rightarrow \text{int}_L(x) \leq \text{int}_L(\text{int}_L(x)). \end{aligned}$$

Thus, $\text{int}_L(x) = \text{int}_L(\text{int}_L(x))$. \square

Corollary 2.2. Let L be a Boolean algebra and c_L satisfies conditions (K1) and (K3). Then $\text{int}_L(x \wedge y) = \text{int}_L(x) \wedge \text{int}_L(y)$.

Proof . It is straightforward by Propositions 1.3 and 2.1. \square

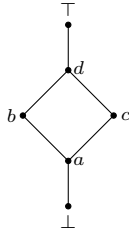
Now let L be a locale and $\text{int}_L: L \rightarrow L$ be a function. Define the function $c_L: L \rightarrow L$ by $c_L(x) = (\text{int}_L(x^*))^*$. It can be shown simply and with the help of De Morgan laws that the following statements are true.

1. If $\text{int}_L(\top) = \top$, then $c_L(\perp) = \perp$.
2. If int_L is an increasing function, then c_L is also increasing.
3. If int_L is a contraction function, then c_L is an expanding function.
4. If $\text{int}_L(x) \wedge \text{int}_L(y) \leq \text{int}_L(x \wedge y)$, then $c_L(x \wedge y) \leq (c_L(x) \vee c_L(y))^{**}$.
5. If int_L is idempotent, increasing and contraction function, then c_L is an idempotent function.

For every $x \in L$, x is *c-closed* if $c_L(x) = x$, and it is *c-open* if $\text{int}_L(x) = x$.

Definition 2.3. Let L be a locale. The function $c_L: L \rightarrow L$ is called an isotonic function on L , if it satisfies the axioms (K0) and (K1).

Example 2.4. Let $L = \{\perp, a, b, c, d, \top\}$ be a locale with the following Hasse diagram.



We define function $c_L: L \rightarrow L$ by $c := \begin{pmatrix} \perp & a & b & c & d & \top \\ \perp & c & d & c & d & \top \end{pmatrix}$. It is easy to see that c_L satisfies (K0), (K1), (K2), (K3) and (K4).

The proof of the following lemma is straightforward.

Lemma 2.5. Let $c_L: L \rightarrow L$ be a function which satisfies (K2). Then for every $x \in L$, $(c_L(x))^\# \leq c_L(x^\#)$ and $\text{int}_L(x) \leq c_L(x)$.

Lemma 2.6. Let L be a locale and $c_L: L \rightarrow L$ be a function such that satisfies (K1). Then $(\text{int}_L(x^\#))^\# \leq c_L(x)$ for every $x \in L$.

Proof . Let $x \in L$, then

$$x^{\#\#} \leq x \Rightarrow c_L(x^{\#\#}) \leq c_L(x) \Rightarrow (c_L(x^{\#\#}))^{\#\#} \leq (c_L(x))^{\#\#} \leq c_L(x).$$

Then $(\text{int}_L(x^\#))^\# = (c_L(x^{\#\#}))^{\#\#} \leq c_L(x)$. \square

Corollary 2.7. Let $c_L: L \rightarrow L$ be a function satisfies condition (K1). Then $\text{int}_L(x^\#) \geq (c_L(x))^\#$.

Proof . It is straightforward. \square

Remark 2.8. Note that if L is a Boolean algebra, then $(\text{int}_L(x^\#))^\# = c_L(x)$ for every $x \in L$.

Corollary 2.9. Let $c_L: L \rightarrow L$ be a function satisfies (K2). Then

1. If $x \in L$ is c-closed, then $x^\#$ is c-open.
2. If L is a Boolean algebra and $x^\#$ is c-open, then x is c-closed.
3. If L is a Boolean algebra and x is c-open, then x^* is c-closed.

Proof . It is straightforward by Remark 2.8. \square

Definition 2.10. Let L be a locale and $c_L: L \rightarrow L$ be a function on L . The *neighborhood function* is a function $\mathcal{N}: L \rightarrow \mathcal{P}(L)$ such that, for every $x \in L$,

$$\mathcal{N}(x) = \{y \in L ; x \leq \text{int}_L(y)\},$$

and the *convergent function* is a function $\mathcal{N}^*: L \rightarrow \mathcal{P}(L)$ defined by

$$\mathcal{N}^*(x) = \{y \in L ; x \leq c_L(y)\}$$

for every $x \in L$.

It is easy to see that if $c_L: L \rightarrow L$ be a function such that satisfies in (K0) and (K2), then $\mathcal{N}^*(\perp) = L$ and $\mathcal{N}(\top) = \{\top\}$.

Proposition 2.11. Let L be a Boolean algebra. Then, the following conditions are equivalent for any isotonic closure function $c_L: L \rightarrow L$.

1. $c_L(a) \wedge b = a \wedge c_L(b) = \perp$.
2. There exist $u \in \mathcal{N}(a)$ and $v \in \mathcal{N}(b)$ such that $a \wedge v = u \wedge b = \perp$.

Proof . (1) \Rightarrow (2) Let $a, b \in L$ and $c_L(a) \wedge b = a \wedge c_L(b) = \perp$. By Proposition 1.3 and Remark 2.8, we have

$$b \leq (c_L(a))^\# = (\text{int}_L(a^\#))^\#\# \leq \text{int}_L(a^\#),$$

that is, $a^\# \in \mathcal{N}(b)$. Thus, there exists $v = a^\# \in \mathcal{N}(b)$ such that $a \wedge v = a \wedge a^\# = \perp$. Similarly, we obtain $a \leq \text{int}_L(b^\#)$, that is, there exists $b^\# \in \mathcal{N}(a)$ with $b^\# \wedge b = \perp$.

(2) \Rightarrow (1) For $a, b \in L$, there exist $u, v \in L$ such that $a \leq \text{int}_L(u)$, $b \leq \text{int}_L(v)$, $a \wedge v = \perp$, and $u \wedge b = \perp$. Since $a \wedge v = \perp$ implies $v \leq a^\#$, we have $b \leq \text{int}_L(v) \leq \text{int}_L(a^\#)$. Hence, $c_L(a) = (\text{int}_L(a^\#))^\# \leq b^\#$. Since b is a complemented element, we conclude that $c_L(a) \wedge b = \perp$. The same argument yields $a \wedge c_L(b) = \perp$. \square

Lemma 2.12. Let L be a Boolean algebra, $\perp \neq x \in L$ and $c_L: L \rightarrow L$ be a function. If $\perp \notin \mathcal{N}^*(x)$, then $c_L(\perp) = \perp$.

Proof . Let $x = c_L(\perp) \neq \perp$, then $x \leq c_L(\perp)$ and so $\perp \in \mathcal{N}^*(x)$. Which is a contradiction, then $c_L(\perp) = \perp$. \square

Proposition 2.13. Let L be a Boolean algebra, $\perp \neq x \in L$ and $c_L: L \rightarrow L$ be a function. Then the following are equivalent

1. $c_L(\perp) = \perp$.
2. $\text{int}_L(\top) = \top$.
3. $\top \in \mathcal{N}(x)$.
4. $\perp \notin \mathcal{N}^*(x)$.

Proof . It is straightforward by Proposition 2.1, Remark 2.8 and Lemma 2.12. \square

Lemma 2.14. Let L be a locale and $c_L: L \rightarrow L$ be a function. Then

1. $(x \leq y \Rightarrow c_L(x) \leq c_L(y))$ if and only if $(y \in \mathcal{N}^*(x), y \leq z \Rightarrow z \in \mathcal{N}^*(x))$.
2. $(x \leq y \Rightarrow \text{int}_L(x) \leq \text{int}_L(y))$ if and only if $(y \in \mathcal{N}(x), y \leq z \Rightarrow z \in \mathcal{N}(x))$,

for all $x, y \in L$.

Proof . (1) *Necessity.* It is straightforward.

Sufficiency. Let $x, y \in L$ with $x \leq y$ be given. It is clear that $x \in \mathcal{N}^*(c_L(x))$. Then $y \in \mathcal{N}^*(c_L(x))$ and so $c_L(x) \leq c_L(y)$.

(2) Similar to (1). \square

Proposition 2.15. Let L be a Boolean algebra. For the function $c_L: L \rightarrow L$ the following statements are equivalent

1. $x \leq y$ implies that $c_L(x) \leq c_L(y)$.
2. $x \leq y$ implies that $\text{int}_L(x) \leq \text{int}_L(y)$.
3. $y \in \mathcal{N}^*(x), y \leq z$ implies that $z \in \mathcal{N}^*(x)$.
4. $y \in \mathcal{N}(x), y \leq z$ implies that $z \in \mathcal{N}(x)$.

Proof . It is straightforward by Lemma 2.14 and Remark 2.8. \square

Proposition 2.16. Let L be a locale, $c_L: L \rightarrow L$ and $x, y \in L$. Then

1. $\text{int}_L(x) \leq x$ if and only if $(y \in \mathcal{N}(x) \Rightarrow x \leq y)$.
2. $x \leq c_L(x)$ if and only if $(x \leq y \Rightarrow y \in \mathcal{N}^*(x))$.

Proof . (1) *Necessity.* It is straightforward.

Sufficiency. It is clear that $x \in \mathcal{N}(\text{int}_L(x))$ and so $\text{int}_L(x) \leq x$. \square

Proposition 2.17. Let $c_L: L \rightarrow L$ be a function and $x, y \in L$. Then, the following statements are equivalent.

1. $c_L(c_L(x)) = c_L(x)$.
2. $y \in \mathcal{N}^*(x)$ if and only if $c_L(y) \in \mathcal{N}^*(x)$.

Proof . (1) \Rightarrow (2). Let $y \in \mathcal{N}^*(x)$. Then $x \leq c_L(y) = c_L(c_L(y))$, which means that $c_L(y) \in \mathcal{N}^*(x)$.

Conversely, let $c_L(y) \in \mathcal{N}^*(x)$. Then $x \leq c_L(c_L(y)) = c_L(y)$ and so $y \in \mathcal{N}^*(x)$.
 (2) \Rightarrow (1). It is clear that $x \in \mathcal{N}^*(c_L(x))$. Then by (2), $c_L(x) \in \mathcal{N}^*(c_L(x))$ and so $c_L(x) \leq c_L(c_L(x))$. On the other hand, $c_L(x) \in \mathcal{N}^*(c_L(c_L(x)))$ and so by (2), $x \in \mathcal{N}^*(c_L(c_L(x)))$. Therefore $c_L(c_L(x)) \leq c_L(x)$. \square

Proposition 2.18. Let $c_L: L \rightarrow L$ be a function and $x, y \in L$. Then, the following statements are equivalent.

1. $\text{int}_L(\text{int}_L(x)) = \text{int}_L(x)$.
2. $y \in \mathcal{N}(x)$ if and only if $\text{int}_L(y) \in \mathcal{N}(x)$.

Proof . It is similar to Proposition 2.17. \square

Proposition 2.19. If $c_L: L \rightarrow L$ is a localic map, then $\mathcal{N}^*(x)$ is a sublocale of L .

Proof . Let $\{y_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{N}^*(x)$. Then $x \leq c_L(y_\lambda)$ for every $\lambda \in \Lambda$ and so $x \leq \bigwedge_{\lambda \in \Lambda} c_L(y_\lambda)$. Since $c_L: L \rightarrow L$ is a localic map, we have $x \leq c_L(\bigwedge_{\lambda \in \Lambda} y_\lambda)$. Therefore $\bigwedge_{\lambda \in \Lambda} y_\lambda \in \mathcal{N}^*(x)$. Since $c_L: L \rightarrow L$ is a localic map, $z \rightarrow y \in \mathcal{N}^*(x)$ for every $y \in \mathcal{N}^*(x)$ and $z \in L$. Then $\mathcal{N}^*(x)$ is a sublocale of L . \square

Definition 2.20. Let $c_L: L \rightarrow L$ be a function on locale L and M be a sublocale of L . Then $c_M: M \rightarrow M$, defined by

$$c_M(x) = \nu_M(c_L(x)) = \bigwedge \{y \in M; c_L(x) \leq y\},$$

is the *relativization* of c_L to M .

Definition 2.21. A property \mathbb{B} of a closure function (L, c_L) is *hereditary* if for every sublocale M of L , (M, c_M) also has the property \mathbb{B} .

Lemma 2.22. The properties (K1), (K2) and (K3) are hereditary properties.

Proof . Let $x, y \in M$ be given.

(K1): If $x \leq y$, then $c_L(x) \leq c_L(y)$. Therefore, $\nu_M(c_L(x)) \leq \nu_M(c_L(y))$ and so by Definition 2.20, $c_M(x) \leq c_M(y)$.

(K2): Since $x \in M$, $\nu_M(x) = x$. Then by condition (K2),

$$x = \nu_M(x) \leq \nu_M(c_L(x)) = c_M(x).$$

(K3): By (K3) we have,

$$\begin{aligned} c_M(x \vee y) &= \nu_M(c_L(x \vee y)) \\ &\leq \nu_M(c_L(x) \vee c_L(y)) \\ &= \nu_M(c_L(x)) \vee \nu_M(c_L(y)) \\ &= c_M(x) \vee c_M(y). \end{aligned}$$

\square

In the following example, we show that the axiom (K4) is not hereditary.

Example 2.23. Let the locale L be as in Example 2.4. Define $c_L: L \rightarrow L$ by $c_L(c) = b$, $c_L(d) = a$ and $c_L(x) = x$ for other elements in L . It is easy to see that c_L satisfies condition (K4). Consider the sublocale $M = \{\perp, c, d, \top\}$ of L . By Definition 2.20, $c_M(c) = d$ and $c_M(d) = c$. Therefore

$$c_M(c_M(c)) = c_M(d) = c \neq c_M(c),$$

which means that, c_M does not satisfy condition (K4).

Proposition 2.24. Let L be a Boolean algebra and let $c_L: L \rightarrow L$ be an isotonic closure function. Then, $\perp \neq x \leq c_L(y)$ if and only if $y^\# \notin \mathcal{N}(x)$.

Proof . *Necessity.* Let $\perp \neq x \leq c_L(y)$ and $y^\# \in \mathcal{N}(x)$. Then, by Remark 2.8 and the definition of neighborhood, $x \leq \text{int}_L(y^\#) \wedge (\text{int}_L(y^\#))^\#$. So $x = \perp$, which is a contradiction. Hence, $y^\# \notin \mathcal{N}(x)$.

Sufficiency. Let $y^\# \in \mathcal{N}(x)$. Then $x \leq \text{int}_L(y^\#) = (c_L(y))^\#$ and so $x \not\leq c_L(y)$. \square

Lemma 2.25. Let L be a Boolean algebra. Then $c_L: L \rightarrow L$ is an isotonic closure function if and only if the neighborhood function $\mathcal{N}: L \rightarrow \mathcal{P}(L)$ satisfies the following conditions.

1. For every $x \in L$, $\top \in \mathcal{N}(x)$.
2. $y \in \mathcal{N}(x)$ and $y \leq z$ imply $z \in \mathcal{N}(x)$.

Proof . *Necessity.* By Proposition 2.1, $\text{int}_L(\top) = \top$ and so, $x \leq \text{int}_L(\top)$ for every $x \in L$. Now, let $y \in \mathcal{N}(x)$ and $y \leq z$. By Proposition 2.1, $\text{int}_L(y) \leq \text{int}_L(z)$. So, $z \in \mathcal{N}(x)$.

Sufficiency. It is clear that by Lemma 2.13 and Proposition 2.15. \square

3 c_L -regular-open and c_L -regular-closed elements

In this section we introduce c_L -regular-closed and c_L -regular-open elements in a locale by closure function $c_L: L \rightarrow L$ and show that the set of all c_L -regular-closed elements of L and the set of all c_L -regular-open elements of L are frames. Also, we define minimal c_L -regular-open element and give conditions where $x \in L$ be a minimal c_L -regular-open.

Definition 3.1. Let L be a locale. An element $a \in L$ is called a **c_L -regular-closed element** if $c_L \text{int}_L a = a$. Also, an element $a \in L$ is called a **c_L -regular-open element** if $\text{int}_L c_L a = a$. The set of all c_L -regular-closed elements of L is denoted by $RC(L)$ and the set of all c_L -regular-open elements of L is denoted by $RO(L)$.

Lemma 3.2. Let L be a locale and $x \in L$. Then, the following statements are true.

1. If $x \in RC(L)$, then $x^\# \in RO(L)$.
2. If $x \in RC(L)$, then $\text{int}_L x \in RO(L)$.
3. If x is c -closed, then $\text{int}_L x$ is c_L -regular-open.
4. If x is c -open, then $c_L x$ is c_L -regular-closed.
5. If x is c -clopen, then $x \in RO(L) \cap RC(L)$.
6. If x is c -open, then $\text{int}_L c_L x$ is c_L -regular-open.
7. If $x \in RO(L)$ and $\text{int}_L: L \rightarrow L$ is idempotent, then x is an c -open element.
8. If $x \in RC(L)$ and $c_L: L \rightarrow L$ is idempotent, then x is a c -close element.

Proof . It is straightforward. \square

Proposition 3.3. Let L be a locale and $c_L: L \rightarrow L$ be a functions such that satisfies in (K1), (K2) and (K4). Then

$$\begin{aligned} RO(L) &= \{\text{int}_L c_L x: x \text{ is } c\text{-open}\} \\ &= \{\text{int}_L y: y \text{ is } c\text{-closed}\}. \end{aligned}$$

and

$$\begin{aligned} RO(L) &= \{c_L \text{int}_L x: x \text{ is } c\text{-close}\} \\ &= \{c_L y: y \text{ is } c\text{-open}\}. \end{aligned}$$

Proof . It is clear by Lemma 3.2. \square

Proposition 3.4. Let L be a locale and $c_L: L \rightarrow L$ be a functions such that satisfies in (K0), (K1), (K2), (K3) and (K4). Then, the following statements are true.

- (1) For every $a \in L$, $c_L \text{int}_L c_L \text{int}_L a = c_L \text{int}_L a$ and $\text{int}_L c_L \text{int}_L c_L a = \text{int}_L c_L a$.
- (2) Let L be a Boolean algebra, $a, b \in L$ and a be c -open. Then

$$c_L(a \wedge b) = c_L(a \wedge c_L b) \quad \text{and} \quad \text{int}_L c_L(a \wedge b) = \text{int}_L c_L a \wedge \text{int}_L c_L b.$$

- (3) Let L be a Boolean algebra, $a, b \in L$ and a be c -closed. Then

$$\text{int}_L(a \vee b) = \text{int}_L(a \vee \text{int}_L b) \quad \text{and} \quad c_L \text{int}_L(a \vee b) = c_L \text{int}_L a \vee c_L \text{int}_L b.$$

- (4) $(RO(L), \leq_L)$ is a frame, $\bigvee_{\lambda \in \Lambda}^{RO(L)} b_\lambda = \bigvee_{\lambda \in \Lambda}^L b_\lambda$ and

$$\bigwedge_{\lambda \in \Lambda}^{RO(L)} b_\lambda = \text{int}_L c_L \text{int}_L \bigwedge_{\lambda \in \Lambda}^L b_\lambda = \text{int}_L c_L \bigwedge_{\lambda \in \Lambda}^L b_\lambda$$

for every $\{b_\lambda\}_{\lambda \in \Lambda} \subseteq RO(L)$.

(5) $(RC(L), \leq_L)$ is a frame, $\bigwedge_{\lambda \in \Lambda}^{RC(L)} b_\lambda = \bigwedge_{\lambda \in \Lambda}^L b_\lambda$ and

$$\bigvee_{\lambda \in \Lambda}^{RC(L)} b_\lambda = c_L \text{int}_L c_L \bigvee_{\lambda \in \Lambda}^L b_\lambda = c_L \text{int}_L \bigvee_{\lambda \in \Lambda}^L b_\lambda$$

for every $\{b_\lambda\}_{\lambda \in \Lambda} \subseteq RC(L)$.

(6) If L is a Boolean algebra, then $(RO(L), \leq_L)$ is a Boolean algebra and $a^* = \text{int}_L a'$ for every $a \in RO(L)$.

(7) If L is a Boolean algebra, $(RC(L), \leq_L)$ is a Boolean algebra and $a^* = c_L a'$ for every $a \in RC(L)$.

Proof . (1). By Proposition 2.1 and condition (K2), $\text{int}_L c_L \text{int}_L a \leq c_L \text{int}_L a$. Then by (K1) and (K4), $c_L \text{int}_L c_L \text{int}_L a \leq c_L c_L \text{int}_L a = c_L \text{int}_L a$. Moreover, $\text{int}_L a \leq c_L \text{int}_L a$. Then, $\text{int}_L a = \text{int}_L \text{int}_L a \leq \text{int}_L c_L \text{int}_L a$ and so $c_L \text{int}_L a \leq c_L \text{int}_L c_L \text{int}_L a$, we conclude that $c_L \text{int}_L c_L \text{int}_L a = c_L \text{int}_L a$. The rest is similar.

(2). Let $a, b \in L$ and a be c -closed. Then by (K1) and (K2),

$$a \wedge b \leq a \wedge c_L b \Rightarrow c_L(a \wedge b) \leq c_L(a \wedge c_L b).$$

Suppose that $\perp \neq a \wedge b$. Let $\perp \neq d \in L$ be c -closed and $c_L(a \wedge c_L b) \leq d$. Then by Corollary 2.7

$$d^\# \leq (c_L(a \wedge c_L b))^\# = \text{int}_L(a \wedge c_L b)^\# \leq (a \wedge c_L b)^\#.$$

Hence, $a \wedge b \leq a \wedge c_L b \leq d$, which implies that $\perp \neq a \wedge b = d \wedge a \wedge b$, and so, $c_L(a \wedge c_L b) \leq c_L(a \wedge b)$.

Now, suppose that $\perp = a \wedge b$. Then

$$b \leq a^* \Rightarrow c_L b \leq a^* \Rightarrow a \wedge c_L b = \perp.$$

Therefore by (K0), $c_L(a \wedge c_L b) = c_L(\perp) = \perp = a \wedge b \leq c_L(a \wedge b)$.

It is evident that $\text{int}_L c_L(a \wedge b) \leq \text{int}_L c_L a \wedge \text{int}_L c_L b$. Also

$$\begin{aligned} \text{int}_L c_L(a \wedge b) &= \text{int}_L c_L \text{int}_L c_L(a \wedge b) = \text{int}_L c_L \text{int}_L c_L(a \wedge c_L b) \\ &\geq \text{int}_L c_L \text{int}_L(a \wedge c_L b) = \text{int}_L c_L(a \wedge \text{int}_L c_L b) \\ &\geq \text{int}_L(c_L a \wedge \text{int}_L c_L b) = \text{int}_L c_L a \wedge \text{int}_L c_L b. \end{aligned}$$

Therefore, $\text{int}_L c_L(a \wedge b) = \text{int}_L c_L a \wedge \text{int}_L c_L b$.

(3). The proof is similar to the proof of part (2).

(4). Let $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq RO(L)$ be given. Let $b \in RO(L)$ with $a_\lambda \leq b$ for every $\lambda \in \Lambda$ be given. Then,

$$\bigvee_{\lambda \in \Lambda} a_\lambda \leq b \Rightarrow \text{int}_{LC_L} \bigvee_{\lambda \in \Lambda} a_\lambda \leq \text{int}_{LC_L} b = b,$$

and since $\bigvee_{\lambda \in \Lambda} a_\lambda \leq \text{int}_{LC_L} \bigvee_{\lambda \in \Lambda} a_\lambda$, we conclude that $\text{int}_{LC_L} \bigvee_{\lambda \in \Lambda} a_\lambda = \bigvee_{\lambda \in \Lambda} a_\lambda \in RO(L)$. Therefore, by part (2), $(RO(L), \leq_L)$ is a frame.

Let $\{b_\lambda\}_{\lambda \in \Lambda} \subseteq RO(L)$ be given. It is clear that $\bigwedge_{\lambda \in \Lambda}^{RO(L)} b_\lambda \leq \bigwedge_{\lambda \in \Lambda}^L b_\lambda$, then

$$\begin{aligned} \bigwedge_{\lambda \in \Lambda}^{RO(L)} b_\lambda \leq \text{int}_L \bigwedge_{\lambda \in \Lambda}^L b_\lambda &\Rightarrow \forall \lambda \in \Lambda \left(\bigwedge_{\lambda \in \Lambda}^{RO(L)} b_\lambda \leq \text{int}_{LC_L} \text{int}_L \bigwedge_{\lambda \in \Lambda}^L b_\lambda \leq \text{int}_{LC_L} \bigwedge_{\lambda \in \Lambda}^L b_\lambda \leq b_\lambda \right) \\ &\Rightarrow \bigwedge_{\lambda \in \Lambda}^{RO(L)} b_\lambda \leq \text{int}_{LC_L} \text{int}_L \bigwedge_{\lambda \in \Lambda}^L b_\lambda \leq \text{int}_{LC_L} \bigwedge_{\lambda \in \Lambda}^L b_\lambda \leq \bigwedge_{\lambda \in \Lambda}^L b_\lambda \\ &\Rightarrow \bigwedge_{\lambda \in \Lambda}^{RO(L)} b_\lambda = \text{int}_{LC_L} \text{int}_L \bigwedge_{\lambda \in \Lambda}^L b_\lambda = \text{int}_{LC_L} \bigwedge_{\lambda \in \Lambda}^L b_\lambda. \end{aligned}$$

(5). The proof is similar to the proof of part (4).

(6). Let $a \in RO(L)$ be given. Then a is complemented in L , which implies that

$$\begin{cases} c_L \text{int}_L a' = c_L (c_L a)' = (\text{int}_{LC_L} a)' = a', \\ \text{int}_{LC_L} \text{int}_L a' = \text{int}_L a' \in RO(L), \\ a \wedge \text{int}_L a' = \text{int}_L (a \wedge a') = \perp, \text{ and} \\ a \vee \text{int}_L a' = \text{int}_{LC_L} (a \vee \text{int}_L a') = \text{int}_L (c_L a \vee c_L \text{int}_L a') = \text{int}_L (c_L a \vee a') \\ \geq \text{int}_L (a \vee a') = \top. \end{cases}$$

Therefore, $(RO(L), \leq_L)$ is a Boolean algebra.

(7). The proof is similar to the proof of part (6). \square

Definition 3.5. A element a of a locale L is said to be:

1. a minimal c_L -regular-open if $\perp \neq x \in RO(L)$ and for every $y \in RO(L)$, $y \leq x$ implies $y = x$ or $y = \perp$.
2. a minimal c_L -regular-closed set if $\perp \neq x \in RC(L)$ and for every $y \in RC(L)$, $y \leq x$ implies $y = x$ or $y = \perp$.
3. a maximal c_L -regular-open if $\top \neq x \in RO(L)$ and for every $y \in RO(L)$, $x \leq y$ implies $x = y$ or $y = \top$.
4. a maximal c_L -regular-closed if $\top \neq x \in RC(L)$ and for every $y \in RC(L)$, $x \leq y$ implies $x = y$ or $y = \top$.

The collection of all minimal c_L -regular-open (resp. minimal c_L -regular-closed , maximal c_L -regular-open , maximal c_L -regular-closed) elements in a locale L is denoted by $MinRO(L)$ (resp. $MinRC(L)$, $MaxRO(L)$, $MaxRC(L)$).

Proposition 3.6. Let x be c -closed and y be minimal c_L -regular-open such that $x < y$, then $\text{int}_L x = \perp$.

Proof . By Lemma 3.2, $\text{int}_L x$ is c_L -regular-open and $\text{int}_L x \leq x \leq y$. Since y is a minimal c_L -regular-open element of L , then $\text{int}_L x = \perp$. \square

Proposition 3.7. Let L be a locale, $x \in L$ and $c_L: L \rightarrow L$ be a function such that satisfies in (K1), (K2) and (K4). Then, the following equivalent:

1. x is minimal c_L -regular-open .
2. $y \leq x \leq c_L y$, where $y \leq x$ and $\text{int}_L y \neq \perp$.
3. $c_L x = c_L y$, where $y \leq x$ and $\text{int}_L y \neq \perp$.

Proof . (1) \Rightarrow (2). Let $y \leq x$ with $\text{int}_L y \neq \perp$ be given. Since $\perp \neq \text{int}_L y \leq \text{int}_L c_L y \leq \text{int}_L c_L x = x$, and x is minimal c_L -regular-open , we conclude that $\text{int}_L c_L y = x$. Therefore, $y \leq x \leq c_L y$.

(2) \Rightarrow (3). It is evident.

(3) \Rightarrow (1). Let z be a c_L -regular-open element such that $\perp \neq z \leq x$. By Lemma 3.2, $\perp \neq \text{int}_L z = z \leq x$, and so by (3), $c_L z = c_L x$. This implies that $z = \text{int}_L c_L z = \text{int}_L c_L x = x$. \square

Proposition 3.8. Let x be minimal c_L -regular-open . If x is c -closed, then x is minimal c_L -regular-closed .

Proof . Since x is c_L -regular-open and c -closed, then x is c_L -regular-closed . Suppose that y is c_L -regular-closed such that $y \leq x$. Then $\text{int}_L y$ is c_L -regular-open and $\text{int}_L y \leq y \leq x$. As x is minimal c_L -regular-open , $\text{int}_L y = \perp$ or $\text{int}_L y = x$. Then either $c_L \text{int}_L y = \perp$ or $c_L \text{int}_L y = c_L x = x$. Since $c_L \text{int}_L y = y$, we get $y = \perp$ or $y = x$. Therefore x is minimal c_L -regular-closed . \square

Proposition 3.9. Let $x \in L$ be c -closed and let $y = c_L \text{int}_L x$. Then y is a c_L -regular-closed element of L and $\text{int}_L y = \text{int}_L x$.

Proof . By Proposition 3.4, $c_L \text{int}_L y = c_L \text{int}_L c_L \text{int}_L x = c_L \text{int}_L x = y$. Then y is a c_L -regular-closed element. Since x is closed, we have $\text{int}_L y = \text{int}_L c_L \text{int}_L x = \text{int}_L c_L \text{int}_L c_L x = \text{int}_L c_L x = \text{int}_L x$. \square

Proposition 3.10. Let $\text{int}_L: L \rightarrow L$ be idempotent, $\top \neq x \in L$ be c -closed and $m \in \text{MaxRO}(L)$ such that $m \leq x$. Then $\text{int}_L x = m$.

Proof . By Lemma 3.2, $m = \text{int}_L m \leq \text{int}_L x$ and $\text{int}_L x$ is a c_L -regular-open, then $\text{int}_L x = m$. \square

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