Journal of Frame and Matrix Theory (JFMT) 1 (2024) No. 2, 37–48
Online ISSN: 3041-8526
Published by Faculty of Mathematics and Computer Science
Hakim Sabzevari University
`http://dx.doi.org/10.22034/jfmt.2024.456708.1011`

Duallity in I-contact algebra

Rahimeh Pourkhandani[a], Mehdi Vatandoost[a,*]

[a] *Faculty of Mathematics and Computer Science, Hakim Sabzevari University, Sabzevar, Iran*

(Communicated by Mostafa Abedi)

Abstract

In topological space, the notions such as convergence and compactness can be characterized in terms of ultrafilters. Also, the concept of clusters, as their counterparts, may be considered as a primitive concept in proximity space. Duality is an alternate way of studying proximity spaces, in which the notion of ends is dual of clusters. In this paper, we define concepts of end and round filter for I-contact algebra and prove that clusters and ends are dual concepts in a similar way to that in the proximity space.

Keywords: I-contact algebra, Proximity space, Cluster, Ultrafilter.
MSC 2020: Primary: 54E05; Secondary: 06D22.

---

1 Introduction

In the early 1950, Efremovich introduced axioms of proximity space (X, δ), where $A\delta B$ means "A is near B" for subsets $A, B \in \mathcal{P}(X)$. The proximity space naturally generalizes the metric space and the topological group [12]. In 1952, Smirnov [13] discovered the connection

---

*Corresponding author
Email addresses: `r.pourkhandani@hsu.ac.ir` (Rahimeh Pourkhandani), `m.vatandoost@hsu.ac.ir` (Mehdi Vatandoost)

Received: 10 May 2024 *Accepted:* 8 July 2024

between the Hausdorff compactification of a Tychonoff space and the compatible proximity relation, showing that “a topological space admits a compatible proximity relation if and only if it is a subspace of a compact Hausdorff space”. Using the ’ends’ of Alexandroff, Smirnov [13] obtained the compactification of a proximity space X, by identifying each point $x \in X$ with the end consisting of all proximity neighbourhoods of x, and showing the compactification of X to be the set of all ends in X.
The concept of a cluster, the analogue in a proximity space of an ultrafilter, was introduced by Leader [10] and provides an alternative approach to many proximity problems. In particular, Leader [10] obtained the compactification of a proximity space X to be the family of all clusters from X. Since the Smirnov compactification is unique, it is evident that a one-to-one correspondence exists between clusters and ends. In fact, Leader [11] proves that clusters and ends are dual classes.

In Euclidean geometry, as a point-based theory the notion of a point is considered as primitive concept, while all other geometrical objects are defined as sets of points. But, in point-free geometry one can consider solid bodies as primitive, and points are defined in terms of these new primitive notions [9]. So, the notion of contact algebra was first introduced by researchers interested in defining point-free geometry. The first relevant attempt in this direction was made by de Laguna [1] in 1922, Whitehead [15] in 1929, Tarski [14], in 1927, Grzegorczyk [8], in 1960, and de Vries [2] in 1962 and many researchers so far. An I-contact algebra is a Boolean algebra on which a proximity relation is defined whith the interpolation property (for short, ICA) were also introduced in [3] by Dimov as normal algebras.

The paper is organized as follows. In Section 2, we review some basic notions together with some known properties of Boolean algebras and I-contact algebras. In Section 3, we introduce duality in contact algebras and prove that the concept of the end is dual of the cluster.

2 Preliminaries

As preliminaries, we recall some definitions and results on contact algebras and topobooleans. For further information, we refer the reader to [3, 4, 5, 6].

We recall from [4, 5] that an algebraic system $(B, C) = (B, \perp, \top, \vee, \wedge, {}', C)$ is called a **contact algebra** (abbreviated as CA) if $(B, \perp, \top, \vee, \wedge, {}')$ is a non-degenerate Boolean algebra and C is a binary relation on B, called **contact**, satisfying the following axioms.

(contact1) aCb implies bCa.

(contact2) $(a \vee b)Cc$ if and only if aCc or bCc.

(contact3) If aCb, then $a \neq \bot$ and $b \neq \bot$.

(contact4) If $a \neq \bot$, then aCa.

Let (B, C) be a contact algebra. The relation $\ll_C$ (or simply $\ll$) of non-tangential inclusion is defined by $a \ll b$ if and only if $a(-C)b'$, i.e., $(a, b') \notin C$; In this case, by inspiring of [12, Definition 3.1], we say that b is a C-neighbourhood of a .The definition of CA can be equivalently reformulated using this relation. This can be done using the following axioms.

($\ll$ 1) $\top \ll \top$.
($\ll$ 2) If $a \ll b$, then $a \leq b$.
($\ll$ 3) It follows from $a \leq b \ll c \leq d$ that $a \ll d$.
($\ll$ 4) $a \ll b$ implies $b' \ll a'$.
($\ll$ 5) $a \ll b$ and $a \ll c$ imply $a \ll (b \wedge c)$.

Also, a contact relation C on a Boolean algebra B which satisfies the interpolation property

(contact5) $x \ll y \Rightarrow (\exists z)(x \ll z \ll y)$,

will be called, for the sake of brevity, **I-contact relation**. If C is an I-contact relation on B, then (B, C) is called an **I-contact algebra** (abbreviated as ICA). Let (B, C) be an I-contact algebra. We recall from [3] that a subset σ of B is called a cluster in (B, C) if and only if the following conditions are satisfied:

(clus1) If a and b belong to σ, then aCb.

(clus2) If aCb for every $b \in \sigma$, then $a \in \sigma$.

(clus3) If $(a \vee b) \in \sigma$, then $a \in \sigma$ or $b \in \sigma$.

Throughout this paper $\mathscr{C}(B, C)$ denote the set of all clusters in (B, C), and for every subset H of B, we define

$$\mathscr{C}_H(B, C) := \{b \in B \colon aCb \text{ for all } a \in H\}.$$

An element a of a bounded lattice L is called an *atom* if $a \neq \bot$, and $\bot < b \leq a$ implies $b = a$ for $b \in L$. In what follows, the set of all atoms of a bounded lattice L is denoted by $At(L)$. Let (B, C) be an I-contact algebra and $a, b \in At(B)$, then (B, C) is called separated if and only if satisfied the following property:

$$aCb \ \ iff \ \ a = b. \tag{2.1}$$

Or equivalently,

$$a \ll b' \ \ iff \ \ a \neq b. \tag{2.2}$$

Lemma 2.1. [7] Let (B, C) be an I-contact algebra. If there exists an $x \in At(B)$ such that aCx and xCb, then aCb for every $a, b \in B$.

Lemma 2.2. [7] Let (B, C) be an I-contact algebra and let $\mathscr{F}$ be an ultrafilter of B. Then $\mathscr{C}_{\mathscr{F}}(B, C)$ is a cluster in I-contact algebra (B, C) such that $\mathscr{F} \subseteq \mathscr{C}_{\mathscr{F}}(B, C)$.

Definition 2.3. Let (B, C) be an I-contact algebra and let $\mathscr{F}$ be an ultrafilter of B. We say that $\mathscr{F}$ generates $\mathscr{C}_{\mathscr{F}}$ or $\mathscr{C}_{\mathscr{F}}$ is determined by $\mathscr{F}$.

Proposition 2.4. [7] Let (B, C) be an ICA. Then, the following statements are true.

(1) A subset σ is a cluster in I-contact algebra (B, C) if and only if there exists an ultrafilter $\mathscr{F}$ in B such that $\sigma = \mathscr{C}_{\mathscr{F}}$.

(2) If σ is a cluster in I-contact algebra (B, C) and $a_0 \in \sigma$, then there exists an ultrafilter $\mathscr{F}$ in B such that $\sigma = \mathscr{C}_{\mathscr{F}}$ and $a_0 \in \mathscr{F}$.

(3) If σ is a cluster in I-contact algebra (B, C) and $\mathscr{F}$ is an ultrafilter in B such that $\mathscr{F} \subseteq \sigma$, then $\sigma = \mathscr{C}_{\mathscr{F}}(B, C)$.

Proposition 2.5. [7] Let (B, C) be an I-contact algebra and $a, b \in B$. Then, the following statements are true.

(1) If $(a, b) \in C$, then there exists a cluster σ in (B, C) such that a and b both belong to σ.

(2) If B is an atomic Boolean algebra and $B_{\tau(C)}$ is a compact topoboolean, then $(a, b) \in C$ if and only if $\mathrm{cl}_{\tau(C)}(a) \wedge \mathrm{cl}_{\tau(C)}(b) \neq \bot$.

3 Some dual concepts in I-contact algebra

Two relation β and β^* on I-contact algebra (B, C) are dual if and only if $a\beta^* b$ is equivalent to $a \not\beta b'$. Clearly $\beta^{**} = \beta$, justifying the term dual. Obviously, the contact relation C and the non-tangential relation $\ll$ are dual relations on B [12]. In this section, we study some dual concepts of this duality such as the cluster and the end.

Definition 3.1. A subset $\mathscr{E}$ in an I-contact algebra (B, C) is called an end in (B, C), if satisfying the following conditions:

(end1) If $a, b \in \mathscr{E}$, then there exists an element d in $\mathscr{E}$ with $d \neq \bot$ such that $d \ll a$ and $d \ll b$.

(end2) If $a \ll b$, then either $a' \in \mathscr{E}$ or $b \in \mathscr{E}$ for every $a, b \in B$.

It is evident that if $\mathscr{E}$ is an end in an I-contact algebra (B, C), then $\top \in \mathscr{E}$ and $\bot \notin \mathscr{E}$.

Definition 3.2. Let (B, C) be an I-contact algebra. A proper filter $\mathscr{F}$ in B is called a round filter, if for each $a \in \mathscr{F}$, there exists an element b in $\mathscr{F}$ such that $b \ll a$.

Example 3.3. Let (B, C) be an I-contact algebra and $a \in At(B)$. Consider

$$\mathscr{N}_a := \big\{ b \in B \colon a \ll b \big\},$$

as the set of all C-neighbourhoods of a, called the C-neighbourhood system at a. We show that this is a round filter in (B, C), as follows:
At first, we prove $\mathscr{N}_a$ is an end in (B, C): If $b, d \in \mathscr{N}_a$, then, by $(\ll 5)$, $a \ll (b \wedge d)$, which implies from (contact5) that there exists an element z in B such that $a \ll z \ll (b \wedge d)$, and so, $\perp \neq z \in \mathscr{N}_a$, $z \ll b$ and $z \ll d$. Let $b, d \in B$ with $b' \notin \mathscr{N}_a$ and $d \notin \mathscr{N}_a$ be given. Then aCb and aCd' and by Lemma 2.1, bCd', and thus $b \not\ll d$.
Furthermore, $\mathscr{N}_a$ is a round filter in (B, C); Because it is, evidently, a filter in B and if $b \in \mathscr{N}_a$, then, by (contact5), there exists an element z in B such that $a \ll z \ll b$, and so, $\perp \neq z \in \mathscr{N}_a$ and $z \ll b$.

Proposition 3.4. Let (B, C) be an I-contact algebra. If $\mathscr{F}$ is a maximal round filter converging to $x \in At(B)$, then $\mathscr{F} = \mathscr{N}_x$.

Proof . Let $\mathscr{F}$ be a maximal round filter converging to $x \in At(B)$(i.e. $\mathscr{N}_x \subseteq \mathscr{F}$). It is sufficient to show that $\mathscr{F} \subseteq \mathscr{N}_x$. If $d \in \mathscr{F}$ then $x \leq d$ or $x \leq d'$. It implies xCd or xCd' and so, $x \ll d$ or $x \ll d'$. Since $d \in \mathscr{F}$ and $d' \notin \mathscr{N}_x \subseteq \mathscr{F}$, we have $x \ll d$ and $d \in \mathscr{N}_x$. □

Proposition 3.5. Let (B, C) be an I-contact algebra. If $\mathscr{E}$ is an end in (B, C), then it is a maximal round filter in (B, C).

Proof . Let $a, b \in \mathscr{E}$ be given. Then, by (end1), there exists an element d in $\mathscr{E}$ with $d \neq \perp$ such that $d \ll a$ and $d \ll b$, which follows from $(\ll 2)$, $\top \in \mathscr{E}$ and $\perp \notin \mathscr{E}$ that $\mathscr{E}$ is dowen directed. Suppose that $(a, b) \in \mathscr{E} \times B$ with $a \leq b$. Then we have

$$\begin{array}{lll}
a \in \mathscr{E} & \Rightarrow \exists d \in \mathscr{E} \setminus \{\perp\}(d \ll a), & \text{by (end1)} \\
& \Rightarrow \exists d \in \mathscr{E} \setminus \{\perp\}(d \ll b), & \text{by } (\ll 3) \text{ and } \ a \leq b \\
& \Rightarrow \exists d \in \mathscr{E} \setminus \{\perp\}(d' \in \mathscr{E} \text{ or } b \in \mathscr{E}), & \text{by (end2)} \\
& \Rightarrow \exists d, e \in \mathscr{E} \setminus \{\perp\}\big((e \ll d \text{ and } e \ll d') \,\text{or}\, b \in \mathscr{E}\big), & \text{by (end1)} \\
& \Rightarrow \exists d, e \in \mathscr{E} \setminus \{\perp\}\big(e \leq d \wedge d' = \perp \text{ or } b \in \mathscr{E}\big), & \text{by } (\ll 2) \\
& \Rightarrow b \in \mathscr{E}. &
\end{array}$$

By the above argument, $\mathscr{E}$ is a proper filter in B. Also, by (end1), $\mathscr{E}$ is round filter in (B, C).

Finally, we must show that the round filter is maximal. Let $\mathscr{R}$ be a round filter such that $\mathscr{E} \subseteq \mathscr{R}$. Then we have

$$\begin{aligned} a \in \mathscr{R} \quad & \Rightarrow \exists d \in \mathscr{R} \setminus \{\perp\}(d \ll a \text{ and } d' \notin \mathscr{R}), && \text{by (end1)} \\ & \Rightarrow \exists d \in \mathscr{R} \setminus \{\perp\}((d' \in \mathscr{E} \text{ or } a \in \mathscr{E}), \text{ and } d' \notin \mathscr{E}), && \text{by (end2)} \\ & \Rightarrow a \in \mathscr{E}. \end{aligned}$$

Hence, $\mathscr{E} = \mathscr{R}$. Therefore, $\mathscr{E}$ is a maximal round filter in (B, C). □

Proposition 3.12 is the converse of Proposition 3.5. Here, we provide the requirements for its proof, as follows:

Definition 3.6. An I-contact algebra (B, C) is called discrete; provided that aCb if and only if $a \wedge b \neq \perp$ for every $a, b \in B$.

Corollary 3.7. If I-contact algebra (B, C) is discrete, then ends coincide with ultrafilters.

Proof . Let $\mathscr{E}$ be an end in (B, C), then, by Proposition 3.5, it is a filter in B. Let $a \in B$ be given. Then $a \wedge a' = \perp$, which implies from definition of C that $a \ll a$, and so, by (end2), $a \in \mathscr{E}$ or $a' \in \mathscr{E}$. Therefore, $\mathscr{E}$ is an ultrafilters in B.

Let $\mathscr{F}$ be an ultrafilters in B. If $a, b \in \mathscr{F}$, then $\perp \neq a \wedge b \in \mathscr{F}$ and by definition of C, $a \wedge b \ll a \wedge b$, which implies from ($\ll$3) that $a \wedge b \ll a$ and $a \wedge b \ll b$, which completes the proof of (end1) for $\mathscr{F}$.

Let $a, b \in B$ with $a \ll b$ be given. Then, by definition of C, $a \wedge b' = \perp$, which proves that $a \notin \mathscr{F}$ or $b' \notin \mathscr{F}$ since it is a filter in B. Because of the maximality of $\mathscr{F}$, $a' \in \mathscr{F}$ or $b \in \mathscr{F}$, which completes the proof of (end2) for $\mathscr{F}$. Therefore, $\mathscr{F}$ is an end in (B, C). □

Definition 3.8. Two subsets V and W of a Boolean algebra B are dual iff $a \in V$ is equivalent to $a' \notin W$.

Definition 3.9. Let (B, C) be an I-contact algebra and $A \subseteq B$.

(1) $A^0 := \{b \in B \colon \text{there exists an element } a \text{ in } A \text{ such that } a \ll b\}$.
(2) $A' := \{b \in B \colon aCb \text{ for every } a \in A\}$.
(3) $A^* := \{b \in B \colon b' \notin A\}$, the dual of A in B.

Lemma 3.10. Let (B, C) be an I-contact algebra. Then, the following statements are true.

(1) For subset $\mathscr{G}$ of B, $(\mathscr{G}^0)^* = \mathscr{G}'$.
(2) For subset $\mathscr{G}$ of B, $(\mathscr{G}')^* = \mathscr{G}^0$.
(3) If $\mathscr{F}$ is an ultrafilter in B, then $\mathscr{F} = \mathscr{F}^*$.

(4) If $\mathscr{F}$ is an ultrafilter in B, then $\mathscr{F} \subseteq \mathscr{C}_{\mathscr{F}}(B,C) = \mathscr{F}'$.

Proof . (1). We have

$$a \in \big(\mathscr{G}^0\big)^* \Leftrightarrow a' \notin \mathscr{G}^0 \Leftrightarrow \forall b \in \mathscr{G}(b \not\ll a') \Leftrightarrow \forall b \in \mathscr{G}(bCa) \Leftrightarrow a \in \mathscr{G}'.$$

(2). We have

$$a \in (\mathscr{G}')^* \Leftrightarrow a' \notin \mathscr{G}' \Leftrightarrow \exists b \in \mathscr{G}\big(b(-C)a'\big) \Leftrightarrow \exists b \in \mathscr{G}(b \ll a) \Leftrightarrow a \in \mathscr{G}^0.$$

From

$$a \in A^0 \Leftrightarrow \exists b \in A(b \ll a) \Leftrightarrow \exists b \in A(a'(-C)b) \Leftrightarrow a' \notin A',$$

we conclude that A^0 and A' are dual subsets of (B,C).

(3). For every $a \in B$, we have

$$a \in \mathscr{F} \Leftrightarrow a' \notin \mathscr{F} \Leftrightarrow a \in \mathscr{F}^*.$$

(4). It is evident. □

Lemma 3.11. Let (B,C) be an I-contact algebra and let $a,b \in B$ with $a \ll b$ be given. If $\mathscr{F}$ is a round filter in (B,C) such that $\mathscr{G} := \big\{a \wedge x \colon x \in \mathscr{F}\big\} \subseteq B \setminus \{\perp\}$, then $\mathscr{G}^0$ is a round filter in (B,C) such that $b \in \mathscr{G}^0$ and $\mathscr{F} \subseteq \mathscr{G}^0$.

Proof . Let $p,q \in \mathscr{G}^0$ be given. Then, by definition, there exist members $v,w \in \mathscr{F}$ such that $a \wedge v \ll p$ and $a \wedge w \ll q$, which implies that $a \wedge v \wedge w \ll p$ and $a \wedge v \wedge w \ll q$, and so, by ($\ll$5), $a \wedge v \wedge w \ll p \wedge q$. Since $\mathscr{F}$ is a filter in B, we infer that $v \wedge w \in \mathscr{F}$, which implies that $a \wedge v \wedge w \in \mathscr{G}$, and so, $p \wedge q \in \mathscr{G}^0$.
Let $(r,s) \in \mathscr{G}^0 \times B$ with $r \leq s$ be given. Then, by definition, there exist member $t \in \mathscr{F}$ such that $a \wedge t \ll r$, which implies form ($\ll$3) that $a \wedge t \ll s$, and so, $s \in \mathscr{G}^0$.

By the above argument, $\mathscr{G}^0$ is a proper filter in B. Suppose that $g \in \mathscr{G}^0$, then, by definition, there exist member $h \in \mathscr{F}$ such that $a \wedge h \ll g$, which implies form (contact5) that there exists an element z in B such that $a \wedge h \ll z \ll g$. An immediate consequence of this last result is that $\mathscr{G}^0$ is a round filter in (B,C). Also, from $a \ll b$, $a \wedge x \ll b$ for every $x \in \mathscr{F}$, then $b \in \mathscr{G}^0$.

Let $f \in \mathscr{F}$ be given. Since $\mathscr{F}$ is a round filter, there exists an element k in $\mathscr{F}$ such that $k \ll f$, which implies that $a \wedge k \ll f$ and $a \wedge k \in \mathscr{G}$, and so, $f \in \mathscr{G}^0$. Therefore, $\mathscr{F} \subseteq \mathscr{G}^0$. □

Proposition 3.12. Let (B,C) be an I-contact algebra. Then $\mathscr{F}$ is an end in (B,C) if and only if it is a maximal round filter in (B,C).

Proof . Let $\mathscr{F}$ be a maximal round filter in (B, C). Let $a, b \in \mathscr{F}$ be given. Then, by definition, there exist members $h, k \in \mathscr{F} \setminus \{\perp\}$ such that $h \ll a$ and $k \ll b$, which proves that $h \wedge k \in \mathscr{F} \setminus \{\perp\}$, $h \wedge k \ll a$ and $h \wedge k \ll b$, which completes the proof of (end1). In verifying (end2), suppose $a \ll b$ and $b \notin \mathscr{F}$. If $a \wedge b \neq \perp$ for every $b \in \mathscr{F}$, then, by Lemma 3.11, there exists a round filter $\mathscr{G}$ in (B, C) such that $b \in \mathscr{G}$ and $\mathscr{F} \subsetneqq \mathscr{G}$. This is a contradiction to the fact that $\mathscr{F}$ is maximal. Hence, there exists an element $z \in \mathscr{F}$ such that $a \wedge z = \perp$, which implies that $z \leq a'$ and $a' \in \mathscr{F}$ since $\mathscr{F}$ is a filter, proving (end2). Therefore, $\mathscr{F}$ is an end in (B, C). In view of Proposition 3.5, the proof is now complete. □

Corollary 3.13. Let (B, C) be an I-contact algebra. Every round filter in (B, C) is a subset of some end in (B, C).

Proof . Since every round filter in (B, C) is contained in a maximal round filter in (B, C), in view of Proposition 3.12, the proof is now complete. □

Proposition 3.14. Let (B, C) be an I-contact algebra and $\mathscr{F} \subseteq B$. Then $\mathscr{F}$ is an end in (B, C) if and only if $\mathscr{F}^*$ is a cluster in (B, C).

Proof . *Necessity.* Let $\mathscr{F}$ be an end in (B, C). Suppose that a and b belong to $\mathscr{F}^*$, then, by definition, $a' \notin \mathscr{F}$ and $b' \notin \mathscr{F}$, which implies from (end2) that $a \not\ll b'$, and so, aCb, which completes the proof of (clus1).

Let $a \in B$ with aCb for every $b \in \mathscr{F}^*$ be given. If $a \notin \mathscr{F}^*$, then we have

$$\begin{aligned}
a' \in \mathscr{F} \quad &\Rightarrow \exists d \in \mathscr{F} \setminus \{\perp\}(d \ll a'), && \text{by (end1)}\\
&\Rightarrow \exists d \in \mathscr{F} \setminus \{\perp\}(a \ll d' \text{ and } d \notin \mathscr{F}^*) && \text{by ($\ll$4) and the fact that ``aCb for every $b \in \mathscr{F}^*$''}\\
&\Rightarrow \exists d \in \mathscr{F} \setminus \{\perp\}(d \ll a' \text{ and } d' \in \mathscr{F}), && \text{by definition of } \mathscr{F}^*\\
&\Rightarrow \exists d, e \in \mathscr{F} \setminus \{\perp\}(e \ll d \text{ and } e \ll d'), && \text{by (end1)}\\
&\Rightarrow \exists d, e \in \mathscr{F} \setminus \{\perp\}(e \leq d \wedge d' = \perp), && \text{by ($\ll$5).}
\end{aligned}$$

Thus, $a \in \mathscr{F}^*$, which completes the proof of (clus2).

Let $a, b \in B$ with $(a \vee b) \in \mathscr{F}^*$ and $a \notin \mathscr{F}^*$ be given. Then $(a' \wedge b') \notin \mathscr{F}$ and $a' \in \mathscr{F}$. From

$$\begin{aligned}
b' \in \mathscr{F} \quad &\Rightarrow \exists d \in \mathscr{F} \setminus \{\perp\}(d \ll a' \text{ and } d \ll b'), && \text{by (end1)}\\
&\Rightarrow \exists d \in \mathscr{F} \setminus \{\perp\}\big(d \ll (a \vee b)'\big) && \text{by ($\ll$5)}\\
&\Rightarrow \exists d \in \mathscr{F} \setminus \{\perp\}(d \ll (a \vee b)' \text{ and } d' \in \mathscr{F}), && \text{by (clus1) for } \mathscr{F}^*\\
&\Rightarrow \exists d, e \in \mathscr{F} \setminus \{\perp\}(e \ll d \text{ and } e \ll d'), && \text{by (end1)}\\
&\Rightarrow \exists d, e \in \mathscr{F} \setminus \{\perp\}(e \leq d \wedge d' = \perp), && \text{by ($\ll$5),}
\end{aligned}$$

we conclude that $b \in \mathscr{F}^*$, which completes the proof of (clus3). Therefore, $\mathscr{F}^*$ is a cluster in (B, C).

Sufficiency. Let $\mathscr{F}^*$ be a cluster in (B, C). If $a, b \in B$ with $a \ll b$, then, by (clus1), $a \notin \mathscr{F}^*$ or $b' \notin \mathscr{F}^*$, which shows that $a' \in \mathscr{F}$ or $b \in \mathscr{F}$, which completes the proof of (end2).

Let $a, b \in \mathscr{F}$ be given. Then $a' \notin \mathscr{F}^*$ and $b' \notin \mathscr{F}^*$, which implies from (clus3) that $a' \vee b' \notin \mathscr{F}^*$, which says that $a \wedge b \in \mathscr{F}$. Thus, by (clus2), there exists an element $\perp \neq d \in \mathscr{F}^*$ such that $d \ll (a \wedge b)$. Hence, by (contact5), there exists an element $\perp \neq z$ in B such that $d \ll z \ll (a \wedge b)$, which implies from (end2) for $\mathscr{F}$ that $z \in \mathscr{F}$ since $d' \notin \mathscr{F}$, and so, by ($\ll$3), $z \ll a$ and $z \ll b$, which completes the proof of (end1). Therefore, $\mathscr{F}$ is an end in (B, C). $\square$

Corollary 3.15. If I-contact algebra (B, C) is discrete, then clusters coincide with ultrafilters.

Proof . By Corollary 3.7, Proposition 3.14 and Lemma 3.10, it is evident. $\square$

Corollary 3.16. Let (B, C) be an I-contact algebra and $\mathscr{G} \subseteq B$. Then $\mathscr{G}^0$ is an end in (B, C) if and only if $\mathscr{G}'$ is a cluster in (B, C).

Proof . By Lemma 3.10 and Proposition 3.14, $\mathscr{G}^0$ is an end in (B, C) if and only if $\mathscr{G}'$ is a cluster in (B, C). $\square$

Proposition 3.17. Let (B, C) be an I-contact algebra. Then Every ultrafilter $\mathscr{F}$ in (B, C) contains a unique end $\mathscr{F}^0$.

Proof . Let $\mathscr{F}$ be an ultrafilter in (B, C). Then, by Lemma 2.2, $\mathscr{C}_{\mathscr{F}}$ is a cluster in I-contact algebra (B, C) such that $\mathscr{F} \subseteq \mathscr{C}_{\mathscr{F}}(B, C) = \mathscr{F}'$. Taking duals, we obtain

$$\mathscr{F}^0 = (\mathscr{F}')^* = \mathscr{C}^*_{\mathscr{F}} \subseteq \mathscr{F}^* = \mathscr{F}.$$

That $\mathscr{F}^0 \subseteq \mathscr{F}$ is an end follows from Corollary 3.16.

To prove uniqueness, suppose that $\mathscr{E}$ is any end contained in $\mathscr{F}$. Then, by Proposition 3.14, $\mathscr{E}^*$ is a cluster containing $\mathscr{F}$, and hence equals $\mathscr{C}_{\mathscr{F}} = \mathscr{F}'$. Thus $\mathscr{E} = (\mathscr{F}')^* = \mathscr{F}^0$, by Lemma 3.10. $\square$

Example 3.18. Let $\mathbb{R}$ is equipped with Euclidean topology, and B is the Boolean algebra $\big(\mathcal{P}(\mathbb{R}), \subseteq\big)$. The relation C on $\mathcal{P}(\mathbb{R})$ is defined by ACB if and only if $\overline{A} \cap \overline{B} \neq \emptyset$, as the standard contact relation. For an arbitrary element x in $\mathbb{R}$, consider the following three subsets of $\mathcal{P}(\mathbb{R})$:

$$\begin{aligned}
\mathscr{N}_x &:= \big\{A \subseteq \mathbb{R} \colon \{x\} \ll A\big\} = \big\{A \subseteq \mathbb{R} \colon x \in int(A)\big\},\\
\mathscr{F}_x &:= \big\{A \subseteq \mathbb{R} \colon x \in A\big\},\\
\sigma_x &:= \big\{A \subseteq \mathbb{R} \colon x \in \overline{A}\big\}.
\end{aligned}$$

It can be evidently proved, from definitions, that $\mathscr{N}_x \subsetneq \mathscr{F}_x \subsetneq \sigma_x$, and these are examples of end, ultrafilter and cluster in B, respectively.

Proposition 3.19. Let (B, C) be an I-contact algebra and $a, b \in B$. Then $a \ll b$ in (B, C) if and only if every end in (B, C) contains either a' or b.

Proof . *Necessity.* The assertion results immediately from (end2).

Sufficiency. Suppose $a \not\ll b$ in (B, C), i.e., aCb'. By Propositions 2.4 and 2.5, there exists an ultrafilter $\mathscr{F}$ in B such that the cluster $\mathscr{C}_{\mathscr{F}}(B, C) = \mathscr{F}'$ containing both a and b'. Then, by Proposition 3.14, $\mathscr{F}'^*$ is an end in (B, C) which a' and b are not in $\mathscr{F}'^*$ and this is a contradiction. □

4 Conclusion

The family of all ends of a proximity space was highlighted in some researches, such as Smirnov compactification [12]. Later, it is shown that this compactification can be obtained, as the family of all clusters of the space. Also, an axiomatic characterization of the family of all ends was presented in a separated proximity space. We generalized the Smirnov compactification theorem from proximity spaces to atomic ICA-topobooleans, by clusters [7]. And so, there exists a natural question: "Is there an analogue axiomatic characterization of the family of all ends in ICA-topobooleans?" Through the results of this article, a characterization of the family of all ends in a special ICA (a separated complete ICA) can be suggested as follows:
Let B is a complete Boolean algebra and Φ be a family of filters (called Φ-filters) in B. Then Φ is the family of all ends for some separated complete ICA on B if and only if the following conditions are satisfied:

i) If $\mathscr{F}$ is a Φ-filter, then there exists at most one atom a of B such that $a \leq \bigwedge_{f \in \mathscr{F}} F$ (Proposition 3.5, (end2) and Equation 2.2).

ii) If $\mathscr{F} \in \Phi$ and $b \in \mathscr{F}$, there is an $a \in \mathscr{F}$ such that every Φ-filter contains either a' or b (Proposition 3.19 and (end1)).

iii) If every Φ-filter contains at least one of the elements c or d, then there is an element b such that every Φ-filter which does not contain c contains b, and every Φ-filter which does not contain d contains b' (contact5).

iv) Every ultrafilter in B contains some Φ-filter, as a subclass (Proposition 3.17).

Also, if Φ is a family of filters (of a complete Boolean algebra B) satisfying this four conditions we define:

$$a \ll b \text{ if and only if every } \Phi\text{-filter contains either } a' \text{ or } b.$$

And then, a contact C on B can be defined by the dual relation $\ll$, in which Φ is a family of all ends of the separated complete ICA (B, C).

References

[1] T. de Laguna, *Point, line, and surface, as sets of solids*, J. Philos., **19** (1922) 449–461. doi

[2] H. de Vries, *Compact Spaces and Compactifications: An Algebraic Approach*, Van Gorcum, Assen, 1962. doi

[3] G. Dimov, D. Vakarelov, *Contact algebras and region-based theory of space: a proximity approach. I*, Fund. Inform., **74** (2006) 209–249. zbl MR doi

[4] G. Dimov, D. Vakarelov, *Contact algebras and region-based theory of space: proximity approach. II*, Fundam. Inform., **74** (2006) 251–282. zbl MR doi

[5] A.A. Estaji, T. Haghdadi, and J. Farokhi Ostad, *Topobooleans and Boolean contact algebras with interpolation property*, Filomat, **35** (2021) 2895–2909. MR doi

[6] A.A. Estaji, A. Karimi Feizabad, and M. Zarghani, *The ring of real-continuous functions on a topoframe*, Categ. Gen. Algebr. Struct. Appl., **4** (2016) 75–94. zbl MR doi

[7] A.A. Estaji, R. Pourkhandani, and M. Vatandoost, *Compact ICA-topobooleans and the Smirnov compactification theorem*, submitted, 2024.

[8] A. Grzegorczyk, *Axiomatizability of geometry without points*, Synthese, **12** (1960) 228–235. zbl MR doi

[9] L. Landi, *Existentially Closed Contact Algebras*, Master's thesis, University of Milan, 2018. link

[10] S. Leader, *On clusters in proximity spaces*, Fund. Math., **47** (1959) 205–213. zbl MR doi

[11] S. Leader, *On duality in oroximity spaces*, Proc. Am. Math. Soc., **13** (1962) 518–523. zbl MR doi

[12] S.A. Naimpally, B.D. Warrack, *Proximity Spaces*, Cambridge Univ. Press, Cambridge, 1970. zbl doi

[13] Y.M. Smirnov, *On proximity spaces*, Mat. Sb., **31** (1952) 543–574. link

[14] A. Tarski, *Les fondements de la gèomètrie des corps*, First Polish Mathematical Congress, 1927. link

[15] A.N. Whitehead, *Process and reality: An essay in cosmology*, Cambridge Univ. Press, Cambridge, 1929. zbl