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Operator range sharing and reverse order law for Moore-Penrose inverse in Hilbert C^*-modules

Mehdi Mohammadzadeh Karizaki[a,*]

[a] *Department of Computer Engineering, University of Torbat Heydarieh, Torbat Heydarieh, Iran*

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Abstract

This paper investigates the scenario where two operators share the same range, which has proven to be highly beneficial in practical applications, especially for efficiently computing the Moore-Penrose inverse of specific operators. We establish specific conditions under which the reverse order law holds for Moore-Penrose inverses. Furthermore, we explore the relationship between the ranges and Moore-Penrose inverse of operators involved in a factorization $A = BDC$ within the framework of a Hilbert C^*-module.

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*Corresponding author
Email address: m.mohammadzadeh@torbath.ac.ir (Mehdi Mohammadzadeh Karizaki)

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1 Introduction and preliminaries

Hilbert C^*-modules are extensions of Hilbert spaces that allow inner products to take values in a C^*-algebra instead of the field of real or complex numbers. However, some fundamental properties of inner product spaces no longer hold true in general for inner product C^*-modules. Therefore, when studying these modules, it becomes important to explore the conditions and more general situations in which these properties may arise. The book [5] is a widely recognized and commonly used reference for this topic.

The Moore-Penrose inverse is a subject that has garnered significant attention in matrix theory, ring theory, and operator algebra, with a wide range of applications including control theory, signal processing, and estimation theory. The existence of the Moore-Penrose inverse is particularly intriguing when investigating the structure of noncommutative algebras.

Xu and Sheng [9] demonstrated that a bounded adjointable operator between two Hilbert C^*-modules possesses a bounded Moore-Penrose inverse if and only if the operator has a closed range. Guaranteeing the existence of the Moore-Penrose inverse for product operators and computing it is generally a challenging endeavor.

The operator ranges is a compelling and significant problem that arises in operator theory, particularly in the realm of Fredholm operators and generalized inverses. Several useful articles, including references [2, 3, 4, 7, 8, 9], address this issue.

In this paper, we delve into various problems related to operator ranges in Hilbert C^*-modules. One particular focus is on the case where two operators share the same range. This result has proven to be immensely valuable in practical applications, as it facilitates efficient computation of the Moore-Penrose inverse for specific operators. We establish specific conditions under which the reverse order law holds for Moore-Penrose inverses. Furthermore, we explore the relationship between the ranges and Moore-Penrose inverse of operators involved in a factorization $A = BDC$ within the context of a Hilbert C^*-module. Our approach revolves around the block matrix decomposition of operators, enabling us to express this relationship in terms of the corresponding Moore-Penrose inverse of operators. By employing the same technique, we derive novel results concerning the product of operators and their Moore-Penrose inverses in the infinite-dimensional settings of the Hilbert C^*-module.

Let us fix our notation and terminology. A Hilbert $\mathfrak{A}$-module $\mathcal{X}$ is a right $\mathfrak{A}$- module equipped with an $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathfrak{A}$ such that $\mathcal{X}$ is complete with respect to the induced norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ $(x \in \mathcal{X})$. Throughout the rest of this paper, $\mathfrak{A}$ denotes a C^*-algebra and $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{K}$ denote Hilbert $\mathfrak{A}$-modules. Let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ be the set of operators $A : \mathcal{X} \to \mathcal{Y}$ for which there is an operator $A^* : \mathcal{Y} \to \mathcal{X}$ such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. It is known that any element $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be bounded and $\mathfrak{A}$-linear. We call $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the set of adjointable operators from $\mathcal{X}$ to

$\mathcal{Y}$. For any $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the range and the null space of A are represented by $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively. In case $\mathcal{X} = \mathcal{Y}$, the space $\mathcal{L}(\mathcal{X}, \mathcal{X})$, which is abbreviated to $\mathcal{L}(\mathcal{X})$, is a C^*-algebra.

A closed submodule M of $\mathcal{X}$ is said to be *orthogonally complemented* if $\mathcal{X} = M \oplus M^\perp$, where $M^\perp = \{x \in \mathcal{X} : \langle x, y\rangle = 0 \text{ for any } \ y \in M\}$. If $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ does not have closed range, then neither $\mathcal{N}(A)$ nor $\overline{\mathcal{R}(A)}$ needs to be orthogonally complemented. In addition, if $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $\overline{\mathcal{R}(T^*)}$ is not orthogonally complemented, then it may happen that $\mathcal{N}(A)^\perp \neq \overline{\mathcal{R}(A^*)}$; see [5]. The above facts show that the theory of Hilbert C^*-modules are much different and more complicated than that of Hilbert spaces.

An operator $B \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ is an inner inverse of A if $ABA = A$ holds. In this case, A is said to be inner invertible or relatively regular. It is well known that A is inner invertible if and only if the range of A ($\mathcal{R}(A)$) is closed in $\mathcal{Y}$. The Moore-Penrose inverse of $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is an operator $X \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ that satisfies the Penrose equations:

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

The Moore-Penrose inverse of A exists if and only if the range of A is closed in $\mathcal{Y}$. If the Moore-Penrose inverse of A exists, it is unique and denoted by $A^\dagger$.

If $\theta \subseteq \{1, 2, 3, 4\}$ and X satisfies the equations (i) for all $i \in \theta$, then X is a θ-inverse of A. The set of all θ-inverses of A is denoted by $A\{\theta\}$. In particular, $A\{1, 2, 3, 4\} = \{A^\dagger\}$.

The term "orthogonal projection" will be reserved for an operator A that is self-adjoint and idempotent. Based on the definition of the Moore-Penrose inverse, it can be proven that the Moore-Penrose inverse of an operator (if it exists) is unique. Furthermore, $A^\dagger A$ and $AA^\dagger$ are orthogonal projections into $\mathcal{R}(A^*)$ and $\mathcal{R}(A)$, respectively. It is clear that A is Moore-Penrose invertible if and only if A^* is Moore-Penrose invertible [5, Theorem 3.2]. In this case, we have the following relationships: $(A^*)^\dagger = (A^\dagger)^*$, $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$, $A^* = A^*AA^\dagger$, and $A^\dagger = A^*(AA^*)^\dagger$.

2 Conditions for Reverse Order Law of Moore-Penrose Inverses

In the subsequent theorems, we establish certain conditions that, when satisfied, ensure the validity of the reverse order law for Moore-Penrose inverses. These theorems provide insights into the circumstances under which this law holds, shedding light on the relationships between the Moore-Penrose inverses of composite operators.

Here, we shall state an extensively employed lemma in these theorems.

Lemma 2.1. [6, Proposition 2.7] Let $A \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ and $B, C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be such that $\overline{\mathcal{R}(B)} = \overline{\mathcal{R}(C)}$. Then $\overline{\mathcal{R}(AB)} = \overline{\mathcal{R}(AC)}$.

Theorem 2.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathfrak{A}$-modules. If $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $B \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ have closed ranges, then the following statements are equivalent:

(i) $\mathcal{R}(A) = \mathcal{R}(ABA)$,

(ii) $\mathcal{X} = \mathcal{R}(BA) + \mathcal{N}(A)$,

(iii) There exists an operator B' such that $ABB' = A$. In this case,

$$B' = A\left((ABA)^{\dagger}A + \left(1 - (ABA)^{\dagger}ABA\right)Z\right),$$

where $Z \in \mathcal{L}(\mathcal{X}, \mathcal{R}(A))$.

Moreover, $B'A^{\dagger} \in (AB)\{1,2,3\}$ and $A^{\dagger}B'A^{\dagger} \in (ABA)\{1,2,3\}$.

Proof . $(i) \Rightarrow (ii)$ Suppose that statement (i) holds. Then, there exists $K \in \mathcal{L}(\mathcal{X})$ such that $ABAK = A$. In particular, $\mathcal{R}(1 - BAK) \subseteq \mathcal{N}(A)$. For any $z \in \mathcal{X}$, we can write $z = BAKz + (z - BAKz)$. Hence, $\mathcal{X} = \mathcal{N}(A) + \mathcal{R}(BA)$.
$(ii) \Rightarrow (i)$ Assume that statement (ii) holds. Thus, $\mathcal{R}(A) = A(\mathcal{X}) = A(\mathcal{N}(A)) + A(\mathcal{R}(BA)) = \mathcal{R}(ABA)$.

$(i) \Rightarrow (iii)$ According to what has been proved, if statement (i) holds, then there is $K \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that $ABAK = A$. Notice that the general solution of the operator equation $ABAX = A$ is $X = (ABA)^{\dagger}A + (1 - (ABA)^{\dagger}ABA)Z$, where $Z \in \mathcal{L}(\mathcal{X}, \mathcal{R}(A))$ is arbitrary. So, we can write $K = (ABA)^{\dagger}A + (1 - (ABA)^{\dagger}ABA)Z$ for some $Z \in \mathcal{L}(\mathcal{X})$. Thus, by taking $B' = AK$, we conclude that $ABB' = A$.

Moreover, let $X = B'A^{\dagger}$. Since

$$\begin{aligned} ABXAB &= ABB'A^{\dagger}AB = AA^{\dagger}AB = AB, \\ XABX &= B'A^{\dagger}ABB'A^{\dagger} = B'A^{\dagger}AA^{\dagger} = B'A^{\dagger} = X, \\ (ABX)^* &= (ABB'A^{\dagger})^* = (AA^{\dagger})^* = AA^{\dagger} = ABX. \end{aligned}$$

So, $B'A^{\dagger} \in (AB)\{1,2,3\}$.

Also, since $\mathcal{R}(B'A^{\dagger}) \subseteq \mathcal{R}(B') \subseteq \mathcal{R}(A)$, by taking $X = A^{\dagger}B'A^{\dagger}$ we have

$$\begin{aligned} ABAXABA &= ABAA^{\dagger}B'A^{\dagger}ABA = (AB)AA^{\dagger}(B'A^{\dagger})(ABA) \\ &= ABP_{\mathcal{R}(A)}(B'A^{\dagger})ABA = (AB)(B'A^{\dagger})(AB)A = ABA \\ XABAX &= A^{\dagger}B'A^{\dagger}ABAA^{\dagger}B'A^{\dagger} = A^{\dagger}B'A^{\dagger}(AB)B'A^{\dagger} = A^{\dagger}B'A^{\dagger} = X, \\ ABAX &= ABAA^{\dagger}B'A^{\dagger} = ABP_{\mathcal{R}(A)}(B'A^{\dagger}) = ABB'A^{\dagger} = AA^{\dagger}. \end{aligned}$$

Hence, $(ABAX)^* = ABAX$ and consequently, $A^{\dagger}B'A^{\dagger} \in (ABA)\{1,2,3\}$.

$(iii) \Rightarrow (i)$ It is sufficient to show that $\mathcal{R}(A) \subseteq \mathcal{R}(ABA)$. By putting $K = \Big((ABA)^\dagger T + (1-(ABA)^\dagger ABA)Z\Big)$ we have $ABAK = A$. Now, we get $\mathcal{R}(A) = \mathcal{R}(ABA)$. □

Theorem 2.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathfrak{A}$-modules. If $A \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, $B \in \mathcal{L}(\mathcal{Y},\mathcal{X})$, BA and ABA have closed ranges, and $\mathcal{R}(A^*) = \mathcal{R}(BA)$, then $\mathcal{R}(A) = \mathcal{R}(ABA)$.

Proof . Since $\mathcal{R}(A^*) = \mathcal{R}(BA)$, by multiplying from the right side, we have $\mathcal{R}(AA^*) = \mathcal{R}(A) = \mathcal{R}(ABA)$. □

Example 2.4. Let $H = \ell^2$ be the Hilbert C^*-module, the space of square-summable sequences, equipped with the inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$, where $x = (x_i)$ and $y = (y_i)$ are elements of ℓ^2. Let A be the right shift operator on ℓ^2, defined by $A(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$. Let B be the left shift operator on ℓ^2, defined by $B(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$.

Clearly, $\mathcal{R}(A) = \mathcal{R}(ABA)$. To compute the solution B' in the equation $ABB' = A$, we obtain $B'(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, x_4, \ldots)$.

Here, we present a theorem regarding the range of positive powers of operators, specifically when they form orthogonal complemented submodules.

Theorem 2.5. Suppose $A \in \mathcal{L}(\mathcal{X},\mathcal{Y})$. The following assertions are equivalent:

(i) $\overline{\mathcal{R}(A)}$ and $\overline{\mathcal{R}(A^*)}$ are orthogonal complemented submodules;

(ii) $\overline{\mathcal{R}(|A^*|^\beta)}$ and $\overline{\mathcal{R}(|A|^\alpha)}$ are orthogonal complemented submodules for some $\alpha, \beta > 0$.

(iii) For any Hilbert $\mathfrak{A}$-module $\mathcal{Z}$ and $\mathcal{K}$, and for any $A' \in \mathcal{L}(\mathcal{Z},\mathcal{Y})$ and $A'' \in \mathcal{L}(\mathcal{K},\mathcal{X})$ such that $\mathcal{R}(A') \subseteq \mathcal{R}(A)$ and $\mathcal{R}(A'') \subseteq \mathcal{R}(A^*)$, the equations $A' = AX$ and $A'' = A^*Y$ have solutions for $X \in \mathcal{L}(\mathcal{Z},\mathcal{X})$ and $Y \in \mathcal{L}(\mathcal{K},\mathcal{Y})$.

(iv) For any Hilbert $\mathfrak{A}$-module $\mathcal{Z}$ and $\mathcal{K}$, and for any $A' \in \mathcal{L}(\mathcal{Z},\mathcal{X})$ and $A'' \in \mathcal{L}(\mathcal{K},\mathcal{Y})$ such that $\mathcal{R}(A') \subseteq \mathcal{R}(A)$ and $\mathcal{R}(A'') \subseteq \mathcal{R}(A^*)$, the equations $A' = |A^*|^\beta X$ and $A'' = |A|^\alpha Y$ have solutions for some $\alpha, \beta > 0$, where $X \in \mathcal{L}(\mathcal{Z},\mathcal{X})$ and $Y \in \mathcal{L}(\mathcal{K},\mathcal{Y})$.

(v) There exists an operator $U \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ such that $A = U|A|$ and $U^*U = P_{\overline{\mathcal{R}(A^*)}}$.

Proof . (i) $\Rightarrow$ (ii) According to Lemma 3.3 in [6], we have $\overline{\mathcal{R}(A^*A)} = \overline{\mathcal{R}(A^*)}$. Since A^*A is a positive operator, Proposition 2.9 in [6] implies that $\overline{\mathcal{R}(|A|^\alpha)} = \overline{\mathcal{R}(A^*A)}$. Similarly, the same argument implies that $\overline{\mathcal{R}(|A^*|^\beta)} = \overline{\mathcal{R}(AA^*)}$.

(ii) $\Rightarrow$ (i) By setting $\alpha = \beta = 2$ and applying Lemma 3.3 in [6], it is ensured.

(i) $\Leftrightarrow$ (iii) It follows from the proof of Theorem 3.2 in [2].

(ii) $\Leftrightarrow$ (iv) It follows from the proof of Theorem 3.2 in [2].

(i) $\Leftrightarrow$ (v) It follows from Lemma 2.11 in [6]. □

Theorem 2.6. Suppose that $A \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, $B \in \mathcal{L}(\mathcal{Z},\mathcal{X})$ and $U \in \mathcal{L}(\mathcal{Y},\mathcal{K})$ have closed ranges. Then

(i) If $\mathcal{R}(B) = \mathcal{R}(A^*)$, then $(AB)^\dagger = B^\dagger A^\dagger$.

(ii) If $\mathcal{R}(U^*) = \mathcal{R}(A)$, then $(UA)^\dagger = A^\dagger U^\dagger$.

(iii) If $\mathcal{R}(B) = \mathcal{R}(A^*)$ and $\mathcal{R}(U^*) = \mathcal{R}(A)$, then $(UAB)^\dagger = B^\dagger A^\dagger U^\dagger$.

(iv) If $\mathcal{R}(B) = \mathcal{R}(A^*)$, then $(ABB^*B)^\dagger = B^\dagger (B^*)^\dagger B^\dagger A^\dagger$.

(v) If $\mathcal{R}(B) = \mathcal{R}(A^*)$, then $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$.

(vi) If $\mathcal{R}(B) = \mathcal{R}(A^*)$, then $(A^*ABB^*)^\dagger = (B^*)^\dagger B^\dagger A^\dagger (A^*)^\dagger$.

Proof .(i) Using Lemmata [4, Lemma 2.3] and [4, Lemma 2.4], the orthogonal sums $\mathcal{X} = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$, $\mathcal{Y} = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$, $\mathcal{Z} = \mathcal{R}(B^*) \oplus \mathcal{N}(B)$ the matrix representation of A has the form $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}$ where A_1 is invertible and $A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}$. Also $B = \begin{bmatrix} B_1 & 0 \\ B_3 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}$ and $B^\dagger = \begin{bmatrix} D^{-1}B_1^* & D^{-1}B_3^* \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}$, where $D = B_1^*B_1 + B_3^*B_3$ is invertible. Since $\mathcal{R}(B) = \mathcal{R}(A^*)$ then

$$\begin{aligned} BB^\dagger = A^\dagger A \quad &\Leftrightarrow \quad \begin{bmatrix} B_1 & 0 \\ B_3 & 0 \end{bmatrix} \begin{bmatrix} D^{-1}B_1^* & D^{-1}B_3^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &\Leftrightarrow \quad \begin{bmatrix} B_1D^{-1}B_1^* & B_1D^{-1}B_3^* \\ B_3D^{-1}B_1^* & B_3D^{-1}B_3^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{2.0}$$

The equality (2.0) implies that $B_1D^{-1}B_3^* = 0$ and $B_3D^{-1}B_3^* = 0$. By multiplying B_1^* and B_3^* on the left of the respective equations and adding the results, we obtain the equality $B_3 = 0$.

Furthermore, since $D = B_1^*B_1$ is invertible, it means that B_1 has a left inverse.

On the other hand, since $B_1D^{-1}B_1^* = 1$, we can conclude that B_1 has a right inverse. Therefore, B_1 is invertible.

Therefore $(AB)^\dagger = \begin{bmatrix} (A_1B_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} = B^\dagger A^\dagger$

(ii) By replacing A with A^* and replacing B with U^* in equation (i), we can conclude that the desired results are obtained.

(iii) If $\mathcal{R}(B) = \mathcal{R}(A^*)$, it implies that $\mathcal{R}(AB) = \mathcal{R}(AA^*)$. Consequently, we can conclude that $\mathcal{R}(AB) = \mathcal{R}(U^*)$. Referring to equation (ii), we find $(UAB)^\dagger = (AB)^\dagger U^\dagger$. Given the condition $\mathcal{R}(B) = \mathcal{R}(A^*)$, we can simplify by (i) it further to $(AB)^\dagger = B^\dagger A^\dagger$. As a result, we obtain $(UAB)^\dagger = B^\dagger A^\dagger U^\dagger$.

(iv) If $\mathcal{R}(B) = \mathcal{R}(A^*)$, then $\mathcal{R}(B^*B) = \mathcal{R}(B^*A^*)$, by applying part (i) we have $(ABB^*B)^\dagger = B^\dagger(B^*)^\dagger B^\dagger A^\dagger$.

(v) The assumption $\mathcal{R}(B) = \mathcal{R}(A^*)$ concludes that $\mathcal{R}(BB^\dagger) = \mathcal{R}(A^\dagger A)$, by applying part (i) we have $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$.

(vi) The assumption $\mathcal{R}(B) = \mathcal{R}(A^*)$ concludes that $\mathcal{R}(BB^*) = \mathcal{R}(A^*A)$, by applying part (i) we have $(A^*ABB^*)^\dagger = (B^*)^\dagger B^\dagger A^\dagger(A^*)^\dagger$. □

Theorem 2.7. Suppose that $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $B \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$ and $U \in \mathcal{L}(\mathcal{Y}, \mathcal{K})$ have closed ranges such that $\mathcal{R}(B) = \mathcal{R}(A^*)$. Then

(i) $(1 - AA^\dagger + AB)^{-1} = 1 - A^\dagger A + B^\dagger A^\dagger$

(ii) $(1 - A^\dagger A + B^* A^\dagger)^{-1} = 1 - AA^\dagger + A(B^*)^\dagger$

(iii) $(1 - B^\dagger B + B^\dagger A^*)^{-1} = 1 - BB^\dagger + (A^*)^\dagger B$

Proof . (i)By matrix representations, we have:

$$(1 - AA^\dagger + AB)^{-1} = \begin{bmatrix} A_1B_1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} B_1^{-1}A_1^{-1} & 0 \\ 0 & 1 \end{bmatrix} = 1 - A^\dagger A + B^\dagger A^\dagger.$$

By matrix forms (ii) and (iii) are straightforward. □

Theorem 2.8. Suppose that $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $B \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ have closed ranges such that $\mathcal{R}(B) = \mathcal{R}(A^*)$ and $\|A(A^\dagger - B)\| < 1$. Then the operator $1 - AB$ is invertible.

Proof . Through matrix representations of the proof of Theorem (2.6) , we obtain the following equation:

$$A(A^\dagger - B) = \begin{bmatrix} 1 - A_1B_1 & 0 \\ 0 & 0 \end{bmatrix}$$

Given that $\|A(A^\dagger - B)\| < 1$, it follows that $\|1 - A_1B_1\| < 1$. Consequently, the matrix representation $1 - AB$ demonstrates that:

$$1 - AB = \begin{bmatrix} 1 - A_1B_1 & 0 \\ 0 & 1 \end{bmatrix}$$

This matrix is invertible. □

The following theorem establishes the relationships between the Moore-Penrose inverses of compositions of operators and their adjoints, given certain conditions on the ranges and null spaces of operators.

Theorem 2.9. Suppose that $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $B \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$, and $U \in \mathcal{L}(\mathcal{Y}, \mathcal{K})$. Furthermore, assume that both AB and UA have closed ranges. Then

(i) If $\mathcal{X} = \mathcal{R}(B) + \mathcal{N}(A)$, then $(A^*AB)^\dagger = (AB)^\dagger (A^*)^\dagger$.

(ii) If $\mathcal{Y} = \mathcal{R}(U^*) + \mathcal{N}(A^*)$, then $(UAA^*)^\dagger = (A^*)^\dagger (UA)^\dagger$.

(iii) If $\mathcal{X} = \mathcal{R}(B) + \mathcal{N}(A)$ and $\mathcal{Y} = \mathcal{R}(U^*) + \mathcal{N}(A^*)$, then $(UAA^*AB)^\dagger = (AB)^\dagger (A^*)^\dagger (UA)^\dagger$.

Proof . Given that $\mathcal{X} = \mathcal{R}(B) + \mathcal{N}(A)$, we have $\mathcal{R}(A) = A(\mathcal{R}(B) + \mathcal{N}(A)) = \mathcal{R}(AB)$. Similarly, since $\mathcal{Y} = \mathcal{R}(U^*) + \mathcal{N}(A^*)$, we have $\mathcal{R}(A^*) = \mathcal{R}(A^*U^*)$. The desired results can be obtained by applying implications (i)-(iii) of Theorem 2.6. □

3 Relationships between Ranges and Moore-Penrose Inverse in Factorization $A = BDC$

The following theorems establish various relationships between the ranges and Moore-Penrose inverse of operators involved in a factorization $A = BDC$ in the context of a Hilbert C^*-module.

Theorem 3.1. Let $\mathcal{X}$ be a Hilbert C^*-module, and consider an operator $A \in \mathcal{L}(\mathcal{X})$ that can be factorized as $A = BDC$, where B, C, and D are operators with closed ranges satisfying $\mathcal{R}(D) = \mathcal{R}(B^*)$ and $\mathcal{R}(D^*) = \mathcal{R}(C)$. Then

(i) $\mathcal{R}(A) = \mathcal{R}(B)$

(ii) $\mathcal{R}(A^*) = \mathcal{R}(C^*)$

(iii) $A^\dagger = C^\dagger D^\dagger B^\dagger$

(iv) $A^\dagger B = C^\dagger D^\dagger$

(v) $CA^\dagger B = D^\dagger$

(vi) $D = B^\dagger A C^\dagger$

Proof . (i) Since $\mathcal{R}(D^*) = \mathcal{R}(C)$, we have $\mathcal{R}(D) = \mathcal{R}(DD^*) = \overline{\mathcal{R}(DC)}$. This implies that $\overline{\mathcal{R}(BD)} = \mathcal{R}(BDC) = \mathcal{R}(A)$. Furthermore, under the assumption $\mathcal{R}(D) = \mathcal{R}(B^*)$, we can conclude that $\mathcal{R}(BD) = \mathcal{R}(BB^*) = \mathcal{R}(B)$. Combining these two results, we can deduce that $\mathcal{R}(A) = \mathcal{R}(B)$.

(ii) Since $\mathcal{R}(D) = \mathcal{R}(B^*)$, we have $\mathcal{R}(D^*D) = \overline{\mathcal{R}(D^*B^*)} = \mathcal{R}(D^*)$. Additionally, we can conclude that $\overline{\mathcal{R}(C^*D^*)} = \mathcal{R}(C^*D^*B^*) = \mathcal{R}(A^*)$. On the other hand, the assumption $\mathcal{R}(D^*) = \mathcal{R}(C)$ leads to $\overline{\mathcal{R}(C^*D^*)} = \mathcal{R}(C^*C) = \mathcal{R}(C^*)$. By combining these two results, we can deduce that $\mathcal{R}(A^*) = \mathcal{R}(C^*)$.

(iii) Since A, B, C, and D have closed ranges, we can represent these operators using operator matrices based on complete submodules. The representation of the operator matrix for these operators is as follows: $C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix}$, $D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(D^*) \\ \mathcal{N}(D) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(D) \\ \mathcal{N}(D^*) \end{bmatrix}$, $B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}$ where C_1, D_1 and B_1 are invertible, and $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}$. Since $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(C^*)$, we can express A as follows:

$$A = BB^\dagger A = BB^\dagger BDC = BDC$$

and

$$A = AC^\dagger C = BDCC^\dagger C = BDC \quad \Leftrightarrow \quad A^* = C^\dagger C A^*.$$

Now, let's consider the matrix representation of A. We have:

$$\begin{aligned}
A_1 &= P_{\mathcal{R}(B)} A P_{\mathcal{R}(C^*)} = BB^\dagger AC^\dagger C = AC^\dagger C, \\
A_2 &= (1 - P_{\mathcal{R}(B)}) A P_{\mathcal{R}(C^*)} = (1 - BB^\dagger) AC^\dagger C = 0, \\
A_3 &= P_{\mathcal{R}(B)} A P_{(1-\mathcal{R}(C^*))} = BB^\dagger A(1 - C^\dagger C) = A(1 - C^\dagger C) = 0, \\
A_4 &= (1 - P_{\mathcal{R}(B)}) A (1 - P_{\mathcal{R}(C^*)}) = (1 - BB^\dagger) A (1 - C^\dagger C) = 0.
\end{aligned}$$

In other words, in this case, the representation of A can be given by:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}. \tag{3.1}$$

The factorization $A = BDC$ leads to the following form:

$$\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1 D_1 C_1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.2}$$

Therefore, we have:

$$A_1 = B_1 D_1 C_1. \tag{3.3}$$

This indicates that A_1 is invertible. Therefore, we have $A_1^{-1} = C_1^{-1} D_1^{-1} B_1^{-1}$, which implies that $A^\dagger = C^\dagger D^\dagger B^\dagger$.

Parts (iv), (v), and (vi) are derived using the details provided in the proof of part (iii) and equation (3.2). □

Corollary 3.2. Let $\mathcal{X}$ be a Hilbert C^*-module, and let $A \in \mathcal{L}(X)$ have the factorization $A = BC$, where A, B and C have closed ranges and $\mathcal{R}(C) = \mathcal{R}(B^*)$. Then, the following statements hold:

(i) $\mathcal{R}(A) = \mathcal{R}(B)$,
(ii) $\mathcal{R}(A^*) = \mathcal{R}(C^*)$,
(iii) $A^\dagger = C^\dagger B^\dagger$,
(iv) $A^\dagger B = C^\dagger$,
(v) $CA^\dagger B = CC^\dagger$.

Proof . The proof of these statements follows directly from Theorem 3.1. □

In the following theorem, we present the condition for the range of an operator A and its adjoint to be equal, which we refer to as EP (Equal Range Property).

Theorem 3.3. Let $\mathcal{X}$ be a Hilbert C^*-module, and let $A \in \mathcal{L}(X)$ have the factorization $A = BDC$, where A, B, C, and D have closed ranges satisfying $\mathcal{R}(D) = \mathcal{R}(B^*)$, $\mathcal{R}(D^*) = \mathcal{R}(C)$, and $\mathcal{R}(B) = \mathcal{R}(C^*)$. Then, $\mathcal{R}(A) = \mathcal{R}(A^*)$

Proof . By applying Theorem 3.1 with implications (i) and (ii), and considering the assumption $\mathcal{R}(B) = \mathcal{R}(C^*)$, we obtain the desired result. □ The following corollary immediately follows from this theorem.

Corollary 3.4. Let $\mathcal{X}$ be a Hilbert C^*-module, and let $A \in \mathcal{L}(X)$ have the factorization $A = BC$, where A, B, and C have closed ranges satisfying $\mathcal{R}(C) = \mathcal{R}(B^*)$ and $\mathcal{R}(B) = \mathcal{R}(C^*)$. Then, $\mathcal{R}(A) = \mathcal{R}(A^*)$.

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