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Multi-step scheme for the approximation of the fractional Riccati differential equation

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Abstract

In this study, we introduce a high-order technique based on quadratic interpolation in the constructed subintervals. This schema can be used to obtain highly accurate solutions with convergence order $O(h^{3+\beta})$ for $0 < \beta \leq 1$ and step size h in fractional problem types. More precisely, we apply this proposed approach to construct a numerical algorithm for the fractional Riccati differential equation. The capability and accuracy of the discretization plan are demonstrated with two examples.

Keywords: Fractional Riccati differential equation; Block-by-Block scheme; Quadratic interpolation; Caputo fractional derivative.
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1 Introduction

The Riccati equation is a type of first-order nonlinear ordinary differential equation that has important applications in various fields of mathematics, physics, engineering, and control theory [11, 26]. It was named after the Italian mathematician Jacopo Riccati, who studied this equation in the 18th century [18]. One of the significant applications of the Riccati equation is in control theory, particularly in optimal control and estimation problems. It arises in the study of linear quadratic optimal control and the theory of Kalman filters. The Riccati equation plays a crucial role in determining the optimal control law or the optimal estimation filter in these problems [14].

Fractional calculus constitutes a mathematical discipline that extends the principles of differential and integral calculus to orders beyond the realm of integers. It deals with derivatives and integrals of arbitrary real or complex order, allowing for the analysis of phenomena involving fractional powers or fractional dimensions [16].

The utilization of fractional calculus has been prominent across a spectrum of scientific and engineering domains, encompassing disciplines such as physics, biology, economics, control theory, signal processing, and several others. It offers a powerful mathematical tool to describe and analyze phenomena with long memory, fractal behavior, and anomalous diffusion [25, 24].

Fractional differential equations (FDEs) extend the concept of ordinary differential equations to include fractional order derivatives. They play a crucial role in modeling and understanding complex phenomena exhibiting memory, hereditary properties, and non-local interactions [1, 8, 12]. FDEs have gained significant attention in various scientific disciplines, including physics, engineering, biology, and finance [6, 5].

Fractional differential equations have gained significant attention in recent years due to their ability to model complex systems with memory and long-range interactions. Numerical methods for solving these equations have also been evolving to efficiently tackle the challenges posed by their non-local and non-integer order nature. One of the recent developments in numerical approaches for fractional differential equations is the use of fractional calculus, which extends traditional calculus to non-integer orders. This allows for the formulation of more accurate and efficient numerical schemes tailored to the specific properties of fractional differential equations.

Some popular numerical schemes for solving fractional differential equations include: machine learning-assisted methods [22, 20], reduced order modeling [10, 3], high-order spectral method [28, 27] and uncertainty quantification method [13].

Applications of these numerical methods for fractional differential equations span various fields as computational fluid dynamics, climate modeling, biomedical simulations, materials science, optimization and control, quantum computing simulation and earthquake engineering.

These recent developments in numerical methods and their diverse applications highlight the ongoing efforts to enhance computational techniques for solving complex problems across various fields, ultimately driving innovation and progress in science and technology.

The Riccati differential equation is considered in fractional form with Caputo type as follows

$$\begin{cases} {}_0^cD_x^\beta u(x) = f_2(x)u^2(x) + f_1(x)u(x) + f_0(x), \qquad x \in [0, \mathcal{L}], \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where f_0, f_1 and f_2 are known and continuous functions and ${}_0^cD_x^\beta$ indicates the Caputo fractional derivative operator of the order β with $\beta \in (0,1]$ is given by [16]

$${}_0^cD_x^\beta u(x) = \frac{1}{\Gamma(1-\beta)} \int_0^x (x-s)^{-\beta} u'(s) ds. \tag{1.2}$$

Solving fractional Riccati equations involves extending traditional methods for solving ordinary Riccati equations to the fractional setting. Here are a few methods commonly used for solving fractional Riccati equations:

- Fractional Variational Iteration Method (FVIM): The FVIM is an extension of the variational iteration method to fractional differential equations. It involves constructing an iterative series solution by decomposing the fractional Riccati equation into a series of fractional linear and nonlinear equations. The iterative process is repeated until the desired accuracy is achieved. FVIM has demonstrated successful application in addressing a range of fractional Riccati equations within existing scholarly works [21, 2].

- Numerical Approximation Methods: Various numerical approximation methods can be employed to solve fractional Riccati equations. These include finite difference methods, finite element methods, spectral methods, and collocation methods. By discretizing the fractional derivative operator, the equation can be transformed into a system of algebraic equations that can be solved using standard numerical techniques [23, 21, 15, 19, 7].

- Fractional Differential Transform Method (FDTM): The FDTM is an extension of the differential transform approach to fractional differential equations. It involves transforming the fractional Riccati equation into a series of algebraic equations using fractional differential transforms. The resulting algebraic equations can then be solved numerically to obtain the solution of the fractional Riccati equation [4].

It is critical to emphasize that the selection of a methodology can hinge on the particular characteristics exhibited by the fractional Riccati equation, including factors like the order of the fractional derivative and the structural composition of the coefficients. The applicability and effectiveness of these methods can vary depending on the problem at hand.

The proposed method converts the above expressed fractional problem into a system of algebraic equations. To this end, the relation (1.1) becomes equivalent to the integral equation. Then, by dividing the interval into $2\mathcal{M}$ equal sub-intervals, a discretization of $u(x)$ at every point is obtained. Next, by quadratic interpolation in $[x_0, x_1]$ and substituting it into the integral equation, it gets the first step of the equations system. After that, this technique applies in any arbitrary sub-interval $[x_i, x_{i+1}]$. Finally, using the plan of quadratic Lagrange polynomials, we obtain an overall schema of the nonlinear equations system for the original problem.

The primary strengths of the suggested numerical approach lie in its high-order convergence, computational simplicity, and inherent lack of intricacy. Nonetheless, a notable drawback surfaces in the form of the stringent requirement for the function $f_2(x)u^2(x) + f_1(x)u(x) + f_0(x)$ to four times of continuity, a condition necessary to establish convergence order. This aspect impairs the method's efficacy when confronted with equations featuring non-smooth solutions, thereby limiting its applicability in certain scenarios.

The remainder of this investigation unfolds as delineated below: The method employed is expounded upon in Section 2. Findings from simulations are detailed in Section 3. Lastly, the paper concludes with Section 4.

2 Methodology

In this section, we introduce a high-order scheme for the numerical solution of the problem given in (1.1). To this end, we rewrite the relation (1.1) as follows

$$ {}_{0}^{c}D_{x}^{\beta} u(x) = \mathscr{G}\big(x, u(x)\big), \tag{2.1} $$

where $\mathscr{G}\big(x, u(x)\big) = f_2(x)u^2(x) + f_1(x)u(x) + f_0(x)$. Moreover, the initial value problem (2.1) is equivalent to the integral equation is given as

$$ u(x) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \mathscr{G}\big(s, u(s)\big) ds. \tag{2.2} $$

Now, to create a discrete plan, the domain $[0, \mathcal{L}]$ is split into $2\mathcal{M}$ equal sub-intervals at the points x_n, where $x_n = n\kappa$ with $\kappa = \frac{\mathcal{L}}{2\mathcal{M}}$ and for $n = 0, 1, \ldots, 2\mathcal{M}$. Initially, we calculate the solution for the first two iterations. Employing quadratic interpolation, an approximation of $\mathscr{G}\big(x, u(x)\big)$ in $[x_0, x_1]$ can be obtained as

$$ \mathscr{G}\big(x, u(x)\big) \simeq \frac{2(x - x_{\frac{1}{2}})(x - x_1)}{\kappa^2} \mathscr{G}_0 + \frac{-4(x - x_0)(x - x_1)}{\kappa^2} \mathscr{G}_{\frac{1}{2}} + \frac{2(x - x_0)(x - x_{\frac{1}{2}})}{\kappa^2} \mathscr{G}_1, \tag{2.3} $$

where $x_{\frac{1}{2}} = x_0 + \frac{\kappa}{2}$, $\mathscr{G}_i = \mathscr{G}\big(x_i, u_i\big)$ and $u_i = u(x_i)$. Substituting recent relation into (2.2), we can write the approximation of $u(x)$ in x_1 as follows

$$\begin{aligned} u_1 \simeq u_0 + \frac{1}{\Gamma(\beta)} \int_0^{x_1} (x_1 - s)^{\beta-1} \Big[\frac{2(s - s_{\frac{1}{2}})(s - s_1)}{\kappa^2} \mathscr{G}_0 + \frac{-4(s - s_0)(s - s_1)}{\kappa^2} \mathscr{G}_{\frac{1}{2}} \\ + \frac{2(s - s_0)(s - s_{\frac{1}{2}})}{\kappa^2} \mathscr{G}_1 \Big] ds = u_0 + \theta_0 \mathscr{G}_0 + \theta_{\frac{1}{2}} \mathscr{G}_{\frac{1}{2}} + \theta_1 \mathscr{G}_1, \end{aligned} \tag{2.4}$$

in which

$$\theta_0 = \frac{1}{\Gamma(\beta)} \int_0^{x_1} (x_1 - s)^{\beta-1} \frac{2(s - s_{\frac{1}{2}})(s - s_1)}{\kappa^2} ds, \tag{2.5}$$

$$\theta_{\frac{1}{2}} = \frac{1}{\Gamma(\beta)} \int_0^{x_1} (x_1 - s)^{\beta-1} \frac{-4(s - s_0)(s - s_1)}{\kappa^2} ds, \tag{2.6}$$

$$\theta_1 = \frac{1}{\Gamma(\beta)} \int_0^{x_1} (x_1 - s)^{\beta-1} \frac{2(s - s_0)(s - s_{\frac{1}{2}})}{\kappa^2} ds, \tag{2.7}$$

and the interpolation method is utilized to approximate $\mathscr{G}_{\frac{1}{2}}$ that result in

$$\mathscr{G}_{\frac{1}{2}} \simeq \frac{3}{8} \mathscr{G}_0 + \frac{3}{4} \mathscr{G}_1 - \frac{1}{8} \mathscr{G}_2. \tag{2.8}$$

Consequently, we can establish the scheme for the first step as follows

$$u_1 = u_0 + \big(\theta_0 + \frac{3}{8} \theta_{\frac{1}{2}}\big) \mathscr{G}_0 + \big(\frac{3}{4} \theta_{\frac{1}{2}} + \theta_1\big) \mathscr{G}_1 - \frac{1}{8} \theta_{\frac{1}{2}} \mathscr{G}_2. \tag{2.9}$$

In a similar manner, in order to calculate the value of $u(x)$ at x_2 in $[x_0, x_2]$, we make use of quadratic interpolating functions to estimate $\mathscr{G}\big(x, u(x)\big)$ as follows

$$\mathscr{G}\big(x, u(x)\big) \simeq \frac{(x - x_1)(x - x_2)}{2\kappa^2} \mathscr{G}_0 + \frac{(x - x_0)(x - x_2)}{-\kappa^2} \mathscr{G}_1 + \frac{(x - x_0)(x - x_2)}{2\kappa^2} \mathscr{G}_2, \tag{2.10}$$

and when (2.10) is incorporated into (2.2), the plan in the second step results in

$$u_2 = u_0 + \vartheta_0 \mathscr{G}_0 + \vartheta_1 \mathscr{G}_1 + \vartheta_2 \mathscr{G}_2, \tag{2.11}$$

where

$$\vartheta_0 = \frac{1}{\Gamma(\beta)} \int_0^{x_2} (x_2 - s)^{\beta-1} \frac{(s - s_1)(s - s_2)}{2\kappa^2} ds, \tag{2.12}$$

$$\vartheta_1 = \frac{1}{\Gamma(\beta)} \int_0^{x_2} (x_2 - s)^{\beta-1} \frac{(s - s_0)(s - s_2)}{-\kappa^2} ds, \tag{2.13}$$

and

$$\vartheta_2 = \frac{1}{\Gamma(\beta)} \int_0^{x_2} \frac{(s - s_0)(s - s_2)}{2\kappa^2} ds. \tag{2.14}$$

At this point, we proceed with constructing the scheme aimed at obtaining an approximation for $u(x_{2n+1})$ and $u(x_{2n+2})$. To achieve this objective, we employ the same technique used in the preceding steps, resulting in the following outcome

$$\begin{aligned}
u_{2n+1} &= u_0 + \frac{1}{\Gamma(\beta)} \int_0^{x_{2n+1}} (x_{2n+1}-s)^{\beta-1} \mathscr{G}\big(s,u(s)\big)ds \\
&= u_0 + \frac{1}{\Gamma(\beta)} \left(\int_0^{x_1} (x_{2n+1}-s)^{\beta-1} \mathscr{G}\big(s,u(s)\big)ds + \sum_{i=1}^{n} \int_{x_{2i-1}}^{x_{2i+1}} (x_{2n+1}-s)^{\beta-1} \mathscr{G}\big(s,u(s)\big)ds \right) \\
&\simeq u_0 + \frac{1}{\Gamma(\beta)} \int_0^{x_1} (x_{2n+1}-s)^{\beta-1} \left[\frac{2(s-s_{\frac{1}{2}})(s-s_1)}{\kappa^2} \mathscr{G}_0 + \frac{-4(s-s_0)(s-s_1)}{\kappa^2} \mathscr{G}_{\frac{1}{2}} \right. \\
&+ \left. \frac{2(s-s_0)(s-s_{\frac{1}{2}})}{\kappa^2} \mathscr{G}_1 \right] ds + \frac{1}{\Gamma(\beta)} \sum_{i=1}^{n} \int_{x_{2i-1}}^{x_{2i+1}} (x_{2n+1}-s)^{\beta-1} \left[\frac{(s-s_{2i})(s-s_{2i+1})}{2\kappa^2} \mathscr{G}_{2i-1} \right. \\
&+ \left. \frac{(s-s_{2i-1})(s-s_{2i+1})}{-\kappa^2} \mathscr{G}_{2i} + \frac{(s-s_{2i})(s-s_{2i-1})}{2\kappa^2} \mathscr{G}_{2i+1} \right] ds \\
&= u_0 + \lambda_n^0 \mathscr{G}_0 + \lambda_n^1 \mathscr{G}_1 + \lambda_n^2 \mathscr{G}_2 + \sum_{i=1}^{n} \lambda_n^{2i-1} \mathscr{G}_{2i-1} + \lambda_n^{2i} \mathscr{G}_{2i} + \lambda_n^{2i+1} \mathscr{G}_{2i+1}, \quad 1 \le n \le \mathcal{M}-1,
\end{aligned} \tag{2.15}$$

where

$$\lambda_n^0 = \eta_n^0 + \frac{3}{8}\eta_n^1, \tag{2.16}$$

$$\lambda_n^1 = \frac{3}{4}\eta_n^1 + \eta_n^2, \tag{2.17}$$

$$\lambda_n^2 = -\frac{1}{8}\eta_n^1, \tag{2.18}$$

$$\lambda_n^{2i-1} = \frac{1}{\Gamma(\beta)} \int_{x_{2i-1}}^{x_{2i+1}} (x_{2n+1}-s)^{\beta-1} \frac{(s-s_{2i})(s-s_{2i+1})}{2\kappa^2} ds, \tag{2.19}$$

$$\lambda_n^{2i} = \frac{1}{\Gamma(\beta)} \int_{x_{2i-1}}^{x_{2i+1}} (x_{2n+1}-s)^{\beta-1} \frac{(s-s_{2i-1})(s-s_{2i+1})}{-\kappa^2} ds, \tag{2.20}$$

and

$$\lambda_n^{2i+1} = \frac{1}{\Gamma(\beta)} \int_{x_{2i-1}}^{x_{2i+1}} (x_{2n+1}-s)^{\beta-1} \frac{(s-s_{2i})(s-s_{2i-1})}{2\kappa^2} ds, \tag{2.21}$$

in which

$$\eta_n^0 = \frac{1}{\Gamma(\beta)} \int_0^{x_1} (x_{2n+1}-s)^{\beta-1} \frac{2(s-s_{\frac{1}{2}})(s-s_1)}{\kappa^2} ds, \tag{2.22}$$

$$\eta_n^1 = \frac{1}{\Gamma(\beta)} \int_0^{x_1} (x_{2n+1}-s)^{\beta-1} \frac{-4(s-s_0)(s-s_1)}{\kappa^2} ds, \tag{2.23}$$

$$\eta_n^2 = \frac{1}{\Gamma(\beta)} \int_0^{x_1} (x_{2n+1}-s)^{\beta-1} \frac{2(s-s_0)(s-s_{\frac{1}{2}})}{\kappa^2} ds. \tag{2.24}$$

To calculate of $u(x)$ approximation in x_{2n+2} for $n = 1, 2, \ldots, \mathcal{M} - 1$, we obtain

$$\begin{aligned}
u_{2n+2} &= u_0 + \frac{1}{\Gamma(\beta)} \int_0^{x_{2n+2}} (x_{2n+2} - s)^{\beta-1} \mathscr{G}\big(s, u(s)\big) ds \\
&= u_0 + \frac{1}{\Gamma(\beta)} \sum_{i=0}^{n} \int_{x_{2i}}^{x_{2i+2}} (x_{2n+2} - s)^{\beta-1} \mathscr{G}\big(s, u(s)\big) ds \\
&\simeq u_0 + \frac{1}{\Gamma(\beta)} \sum_{i=0}^{n} \int_{x_{2i}}^{x_{2i+2}} (x_{2n+2} - s)^{\beta-1} \bigg[\frac{(s - s_{2i+1})(s - s_{2i+2})}{2\kappa^2} \mathscr{G}_{2i} \qquad (2.25) \\
&+ \frac{(s - s_{2i})(s - s_{2i+2})}{-\kappa^2} \mathscr{G}_{2i+1} + \frac{(s - s_{2i})(s - s_{2i+1})}{2\kappa^2} \mathscr{G}_{2i+2} \bigg] ds \\
&= u_0 + \sum_{i=0}^{n} \mu_n^{2i} \mathscr{G}_{2i} + \mu_n^{2i+1} \mathscr{G}_{2i+1} + \mu_n^{2i+2} \mathscr{G}_{2i+2}, \qquad (2.26)
\end{aligned}$$

in which

$$\mu_n^{2i} = \frac{1}{\Gamma(\beta)} \int_{x_{2i}}^{x_{2i+2}} (x_{2n+2} - s)^{\beta-1} \frac{(s - s_{2i+1})(s - s_{2i+2})}{2\kappa^2} ds, \qquad (2.27)$$

$$\mu_n^{2i+1} = \frac{1}{\Gamma(\beta)} \int_{x_{2i}}^{x_{2i+2}} (x_{2n+2} - s)^{\beta-1} \frac{(s - s_{2i})(s - s_{2i+2})}{-\kappa^2} ds, \qquad (2.28)$$

and

$$\mu_n^{2i+2} = \frac{1}{\Gamma(\beta)} \int_{x_{2i}}^{x_{2i+2}} (x_{2n+2} - s)^{\beta-1} \frac{(s - s_{2i})(s - s_{2i+1})}{2\kappa^2} ds. \qquad (2.29)$$

Hence, by employing the relations (2.9), (2.11), (2.15) and (2.25), we derive a system of equations, the solution of which yields the approximate values of the unknown function $u(x)$ at discrete points x_n for $n = 1, 2, \ldots, \mathcal{M}$. Note that the system of nonlinear equations is solved by the command "fsolve" in Maple 18 software.

Let us now focus on obtaining the truncation errors associated with the discrete scheme (2.9), (2.11), (2.15) and (2.25).

Theorem 2.1. [9] Let $\mathscr{G}\big(x, u(x)\big) \in C^4([0, \mathcal{L}])$ and $\tilde{u}(x_j)$ is the approximation of $u(x_j)$ for $j = 2n + 1, 2n + 2$ with $n = 1, 2, \ldots, \mathcal{M}$ and step size κ. Then, we have

$$|e_j| = |u(x_j) - \tilde{u}(x_j)| = \mathcal{O}(\kappa^{3+\beta}), \qquad 0 < \beta \leq 1. \qquad (2.30)$$

3 Numerical experiments

In this segment, we have utilized the approach introduced to analyze several examples. We have assessed the precision of the outcomes using the formulas presented as follows

$$\ell_\infty = \|u(x) - \tilde{u}(x)\|_\infty = \max_{0 \leq n \leq \mathcal{M}} |u(x_n) - \tilde{u}(x_n)|, \qquad (3.1)$$

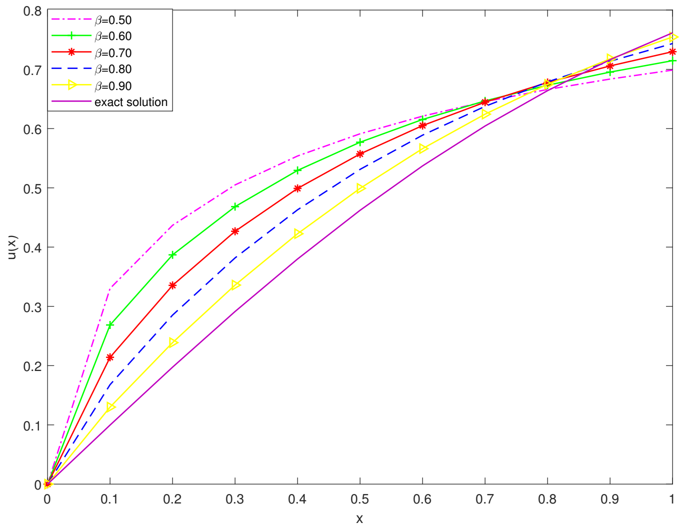

Figure 1: Graphs of approximate solution with $\mathcal{M} = 5\,(\kappa = 0.1)$ on interval $[0, 1]$ for different values of β in Example 3.1.

where $u(x)$ and $\tilde{u}(x)$ are true and numerical solutions, respectively. Moreover, the below relation is utilized to determine the rate of convergence as

$$c - order = \log_2\left(\frac{\mathcal{E}_1}{\mathcal{E}_2}\right), \tag{3.2}$$

in which $\mathcal{E}_1$ and $\mathcal{E}_2$ are ℓ_∞ errors depend on steps κ and $\frac{\kappa}{2}$, respectively.

Example 3.1. At the outset, we analyze our methodology for the principal problem denoted by equation (1.1) as follows

$$\begin{cases} {}_0^cD_x^\beta u(x) = -u^2(x) + 1, \qquad x \in [0, \mathcal{L}], \\ u(0) = 0, \end{cases} \tag{3.3}$$

in which the analytical solution is $u(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1}$ for $\beta = 1$. To solve this model, we employ our approach by varying the parameters $\mathcal{M}$ and β. An analysis of the results is conducted and the corresponding numerical findings are reported in Table 1 in which the investigation focuses on with $\beta = 1$ and $\mathcal{L} = 1, 2$. The convergence rate of the results presented in Table 1 exhibits a noticeable pattern, revealing that the decay rates of the errors are consistently approximate to 4.

Table 1: The ℓ_∞ error and the c-order produced of the yielded outcomes for some values of $\mathcal{M}$ with $\beta = 1$ for Example 3.1.

	$\mathcal{L} = 1$		$\mathcal{L} = 2$	
$\mathcal{M}$	ℓ_∞	c-order	ℓ_∞	c-order
10	$1.9690e-07$	–	$7.7610e-06$	–
20	$8.4586e-09$	4.5409	$2.6101e-07$	4.8940
40	$5.0961e-10$	4.0529	$8.6365e-09$	4.9175
80	$3.1288e-11$	4.0257	$5.0961e-10$	4.0829

This observation aligns remarkably well with the theoretical predictions, indicating a strong concurrence between the empirical findings and the numerical outcomes.

Table 2: Comparison of the ℓ_∞ error of the yielded outcomes and [17] for some values of $\mathcal{M}$ with $\beta = 1$ and $\mathcal{L} = 1$ for Example 3.1.

$N = 2\mathcal{M}$	Proposed method	[17]
10	$5.73e-06$	$1.91e-05$
20	$1.96e-07$	$9.47e-07$
40	$8.45e-09$	$3.43e-08$
80	$5.09e-10$	$1.16e-09$

Also, Table 2 shows a comparison of the ℓ_∞ errors between our plan at $\mathcal{L} = 1$ and method in [17] when $\beta = 1$ in which apparent that the block-by-block plan exhibits superior accuracy in addressing this particular problem. In order to verify the convergence of the problem in fractional mode, Figure 1 displays the numerical solutions for various values of β with $\mathcal{M} = 5$ on the interval $[0, 1]$. The depicted results provide compelling evidence of the convergence of the numerical solutions towards the exact solution as β approaches 1. This alignment between the numerical and exact solutions further supports the validity and accuracy of the computational method employed.

Example 3.2. Consider the problem (1.1) as follows

$$ {}_0^cD_x^\beta u(x) = -u^2(x) + 2u(x) + 1, \tag{3.4} $$

with the homogeneous initial condition in the interval $[0, 1]$. Also, the corresponding true solution is $u(x) = 1 + \sqrt{2}\tanh\left(\sqrt{2}x + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right)$ for $\beta = 1$. We have utilized the

Table 3: The ℓ_∞ error and the c-order produced of the yielded outcomes for some values of κ with $\beta = 1$ for Example 3.2.

κ	ℓ_∞	c-order	time(s)
0.1	$9.0065e-05$	$-$	0.672
0.05	$4.5434e-06$	4.3091	0.890
0.025	$2.4907e-07$	4.1891	2.250
0.0125	$1.4471e-08$	4.1053	9.109

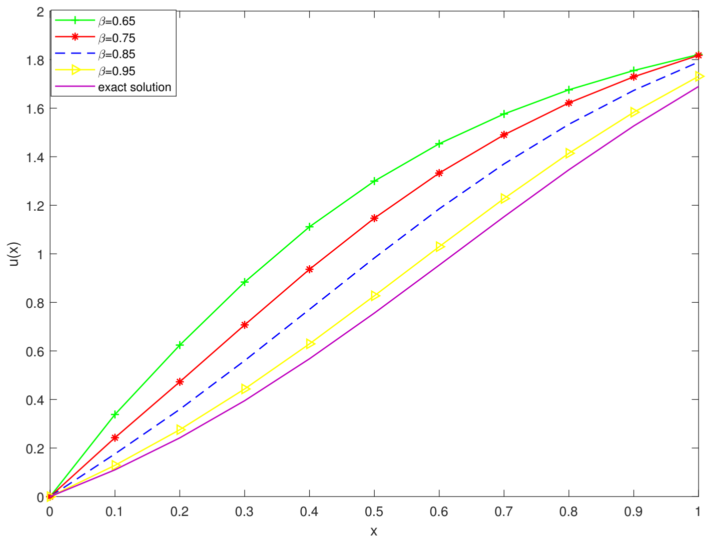

Figure 2: Graphs of approximate solution with $\mathcal{M} = 5$ $(\kappa = 0.1)$ on interval $[0, 1]$ for different values of β in Example 3.2.

proposed technique to address this problem by considering various values of κ and $\beta = 1$. The results obtained using this method within the interval $[0, 1]$ are presented in Table 3. The observed convergence order of this algorithm is approximately $\mathcal{O}(\kappa^4)$, which aligns with the findings stated in Theorem 2.1. Additionally, Figure 2 illustrates the behavior of the obtained results with $\mathcal{M} = 5$. The combination of the outcomes presented in Table 3 and the insights provided by Figure 2 clearly demonstrate the suitability of the reported results. Moreover, these findings highlight the effectiveness and proficiency of the established method in accurately computing the numerical solution for this particular example.

4 Conclusion

In this work, we have introduced a novel schema of $3+\beta$ order to solve Riccati fractional differential equations. Our approach, based on quadratic Lagrange polynomials, follows a block-by-block methodology and provides an error estimate for the numerical solution with an order of $3+\beta$, where $0<\beta\leq 1$. Through extensive numerical tests, the accuracy of the proposed technique is successfully verified. The outcomes derived from solving two specific numerical tests have demonstrated the exceptional precision and efficacy of the developed scheme in tackling such problems.

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