Journal of Frame and Matrix Theory (JFMT) 1 (2023) No. 2, 73–87
Online ISSN: 3041-8526
Published by Faculty of Mathematics and Computer Science
Hakim Sabzevari University
http://dx.doi.org/10.22034/jfmt.2024.462762.1015

A brief overview on phase retrieval problem

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(Communicated by Mehdi Zaferanieh)

Abstract

Phase retrieval is a fundamental problem in signal processing and imaging, where the goal is to recover a signal from its magnitude measurements. This overview paper provides a brief survey of the history, theory and algorithms of phase retrieval topic. We discuss the importance of phase retrieval frames in various fields and highlight key milestones in its development.

Keywords: Phase retrieval, Signal recovery, Measurements, Frames.
MSC 2020: Primary: 30D05; Secondary: 42B10, 42C15.

1 Introduction and background

The phase retrieval problem is a fundamental challenge in engineering and mathematics that plays a crucial role in various fields such as imaging, signal processing, X-ray, electron

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Received: 17 May 2024 *Accepted:* 19 June 2024

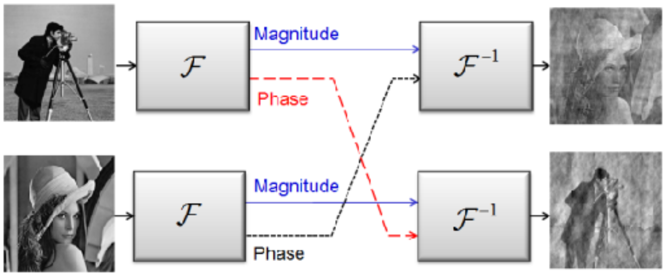
microscopy, optics and much more [10, 27, 37, 38, 40, 45, 46, 47]. The concept of phase retrieval refers to the process of recovering the phase information of a signal or wave from its magnitude measurements. In many practical applications, only the magnitude of a signal can be easily measured, while the phase information is often lost or inaccessible. For instance, in optical settings, detection devices such as photodetectors or cameras, due to several reasons including limited sensitivity and nonlinear response are typically designed to measure the intensity of light rather than the phase of the light wave. Therefore, the ability to accurately retrieve the phase information from the magnitude measurements is essential for reconstructing the original signal or wave. Indeed, by exact retrieving the phase information, engineers can improve the quality of images, and achieve better signal processing results. Hence, phase retrieval is an essential topic in engineering for a wide range of applications.

The history of phase retrieval in mathematics is rich and diverse, with many important milestones and breakthroughs. In fact, mathematical study of phase retrieval dates back to the early 20th century and has been extensively studied in the fields of harmonic analysis, optimization, and signal processing. Over the years, researchers have developed various algorithms and techniques for solving the phase retrieval problem, ranging from iterative algorithms based on convex optimization to non-convex optimization methods and deep learning approaches. In mathematical language the problem of recovery without using phase was presented in [39] as follows: Suppose $x = \{x_i\}_{i=1}^m$ is a vector with non-zero components, the problem is finding x by having the Fourier magnitude-square measurements of x at hand, i.e.;

$$\left|\sum_{n=0}^{m-1} x_n e^{-2\pi i k n/m}\right|^2 \quad 1 \leq k \leq m. \tag{1.1}$$

On the other hand, recovering an image without using true phase information is a difficult task, as the phase information contains crucial structural details of the image. To observe this fact we use Figure 1 of [42], which shows the results of recovering two images by altering their phase information. More precisely, in Figure 1, the Fourier transformation of two images is conducted, followed by the exchange of their Fourier phases, and subsequent inverse Fourier transformation. The outcome distinctly illustrates the significance of Fourier phase.

One of the earlier and most well-known approaches in phase retrieval is the Gerchberg-Saxton algorithm [30], proposed in the 1970s for image reconstruction in electron microscopy. This algorithm, which aims to minimize a non-convex problem associated to 1.1, laid the foundation for many subsequent developments in phase retrieval algorithms and has been widely used in various applications. See [17, 28, 36] for other methods and extensions.



The ongoing research in phase retrieval continues to advance our understanding of this important problem and develop new algorithms and techniques for solving it effectively. These aspects are closely related to frame theory. In what follows, we briefly review some basic definitions and results in frame theory.

Frames in a separable Hilbert space are collections of vectors in underlying space which allow for a stable but not necessarily unique decomposition of an arbitrary element into an expansion of the frame elements and first were introduced in [26]. Indeed, frame analyzes the stability, completeness, and redundancy of linear discrete signal representation.

Definition 1.1. A family of vectors $\Phi := \{\phi_i\}_{i\in I}$ in $\mathcal{H}$ is called a *frame* if there exist the constants $0 < A_\Phi \leq B_\Phi < \infty$ such that

$$A_\Phi\|f\|^2 \leq \sum_{i\in I} |f, \phi_i\rangle|^2 \leq B_\Phi\|f\|^2, \qquad (f \in \mathcal{H}). \tag{1.2}$$

The constants A_Φ and B_Φ are called *frame bounds.* The sequence $\{\phi_i\}_{i\in I}$ is said to be a *Bessel sequence* whenever the right hand side of (1.2) holds. A frame $\{\phi_i\}_{i\in I}$ is called *A-tight frame* if $A = A_\Phi = B_\Phi$, and in the case of $A_\Phi = B_\Phi = 1$ it is called a *Parseval frame.* Given a frame $\Phi = \{\phi_i\}_{i\in I}$, the *frame operator* is defined by $S_\Phi f = \sum_{i\in I}\langle f, \phi_i\rangle\phi_i$. It is a bounded, invertible, and self-adjoint operator [25]. Also, the *synthesis operator* $T_\Phi : l^2 \to \mathcal{H}$ is defined by $T_\Phi\{c_i\} = \sum_{i\in I} c_i\phi_i$. The frame operator can be written as $S_\Phi = T_\phi T_\phi^*$ where $T_\Phi^* : \mathcal{H} \to l^2$, the adjoint of T, given by $T_\Phi^* f = \{\langle f, \phi_i\rangle\}_{i\in I}$ is called the *analysis operator.* It is proved that a family of vectors in a Hilbet space is a frame if and only if the synthesis operator is surjective and so frames span the space. The family $\{S_\Phi^{-1}\phi_i\}_{i\in I}$ is also a frame for $\mathcal{H}$, called the *canonical dual* frame. In general, a frame $\{\psi_i\}_{i\in I} \subseteq \mathcal{H}$ is called an *alternate dual* or simply a *dual* for $\{\phi_i\}_{i\in I}$ if $f = \sum_{i\in I}\langle f, \psi_i\rangle\phi_i$, for $f \in \mathcal{H}$. All frames have at least a dual, the canonical dual, and redundant frames have an infinite number of alternate dual frames. It is known that every dual frame is of the form $\{S_\Phi^{-1}\phi_i + u_i\}_{i\in I}$, where $\{u_i\}_{i\in I}$ is a

Bessel sequence that satisfies $\sum_{i\in I}\langle f,u_i\rangle\phi_i=0$, for all $f\in\mathcal{H}$. Also recall that, two frames Φ and Ψ are equivalent if there exists an invertible operator U on $\mathcal{H}$ so that $\Psi=U\Phi$.

The excess of a frame Φ is defined as the greatest integer k such that k elements can be removed from the frame elements and still leave a complete set, or ∞ if there is no upper bound to the number of elements that can be removed [11, 13]. We denote the excess of a frame Φ by $E(\Phi)$. See [25, 33] for more detailed information on frame theory and [3, 4, 19, 26, 34, 35, 41] for the importance of duality principle.

In a finite dimensional Hilbert space, frames are exactly spanning family of vectors in underlying space. And by a full spark frame $\Phi=\{\phi_i\}_{i\in I_m}$ in a finite dimensional Hilbert space $\mathcal{H}_n$ we mean every Size-n subcollection of Φ spans the space $\mathcal{H}_n$. The concept of full spark frame is essential in signal processing and many other application fields, and specially is important in the theory of phase retrieval (see [2]).

2 Recent perspectives

In 2006, Balan, Casazza and Edidin [16] reformulated phase retrieval problem in a new point of view. Indeed, let $\mathcal{H}$ be a separable Hilbert space and I a countable index set, considering the nonlinear mapping

$$\mathbb{M}_\Phi:\mathcal{H}\to l^2(I),\qquad \mathbb{M}_\Phi(f)=\{|\langle f,\phi_i\rangle|^2\}_{i\in I} \tag{2.1}$$

obtained by taking the absolute value element wise of the analysis operator. Let us denote by $\hat{\mathcal{H}}=\mathcal{H}/\sim$ considered by identifying two vectors which are different in a phase factor, i.e., $f\sim g$ whenever there exists a scalar θ with $|\theta|=1$ so that $g=\theta f$. Obviously in a real Hilbert space we have $\hat{\mathcal{H}}=\mathcal{H}/\{1,-1\}$ and in the complex case $\hat{\mathcal{H}}=\mathcal{H}/\mathbb{T}$, where $\mathbb{T}$ is the complex unit circle. So, the mapping $\mathbb{M}_\Phi$ can be extended to $\hat{\mathcal{H}}$ as $\mathbb{M}_\Phi(\hat{f})=\{|\langle f,\phi_i\rangle|^2\}_{i\in I}$, $f\in\hat{f}=\{g\in\mathcal{H}:g\sim f\}$. The injectivity of the nonlinear mapping $\mathbb{M}_\Phi$ leads to the reconstruction of every signal in $\mathcal{H}$ up to a constant phase factor from the modulas of its frame coefficients. In [16], the authors investigated the injectivity of $\mathbb{M}_\Phi$ in finite dimensional real Hilbert spaces and moreover, it was proven that $4n-2$ measurements suffice for the injectivity in n-dimensional complex Hilbert spaces. Indeed, the injectivity of the mapping $\mathbb{M}_\Phi$ is equivalent to the following definition:

Definition 2.1. A family of vectors $\Phi=\{\phi_i\}_{i\in I}$ in $\mathcal{H}$ does phase retrieval if whenever $f,g\in\mathcal{H}$ satisfy

$$|\langle f,\phi_i\rangle|=|\langle g,\phi_i\rangle|,\qquad (i\in I) \tag{2.2}$$

then there exists a scalar θ with $|\theta|=1$ so that $f=\theta g$.

This aspect is closely related to frame theory and has been a popular topic for many of researchers. And has been extended to some other concepts such as weak phase retrieval and norm retrieval, see [1, 6, 32].

2.1 Finite dimensional case

Suppose that $\Phi = \{\phi_i\}_{i\in I_m}$ is a collection of vectors in a finite dimensional Hilbert space $\mathcal{H}_n$. Balan, Casazza and Edidin [16] proved that if $m \geq 2n-1$, then for a generic frame Φ in $\mathbb{R}^n$ the mapping M_Φ is injective, where by generic we mean an open and dense subset of the set of all m-element frames in $\mathbb{R}^n$. More precisely, a subset $A \subseteq \mathbb{R}^n$ is called generic whenever there exists a non-zero polynomial $p(x_1, ..., x_n)$ so that

$$A^c \subseteq \{(x_1, ..., x_n) \in \mathbb{R}^n : p(x_1, ..., x_n) = 0\}.$$

By a generic set in complex case $\mathbb{C}^n$ we mean that it is generic in $\mathbb{R}^{2n}$.

Proposition 2.2. [16] If $\Phi = \{\phi_i\}_{i\in I_m}$ does phase retrieval in $\mathbb{R}^n$, then $m \geq 2n-1$. If $m \geq 2n-1$ and Φ is full spark then ϕ does phase retrieval. Moreover, $\{\phi_i\}_{i\in I_{2n-1}}$ does phase retrieval if and only if Φ is full spark.

Moreover, the same result was shown in $\mathbb{C}^n$ for $m \geq 4n-2$, see [16]. Then the authors in [23] posed the $4n-4$-Conjecture in complex case $\mathbb{C}^n$ as follows:

Conjecture. Consider the collection $\Phi = \{\phi_i\}_{i\in I_m}$ in $\mathbb{C}^n$. If $n \geq 2$ then the following hold:

(i) If $m < 4n-4$, then M_Φ is not injective.

(ii) If $m \geq 4n-4$, then for a generic frame Φ the mapping M_Φ is injective.

Then, it was shown that the $4n-4$ Conjecture is true when $n = 2, 3$. However, the problem is open in general case.

A linear operator was defined in [14], which presents an equivalent condition for injectivity of the non-linear map $\mathbb{M}_\Phi$, defined by (2.1). To this end, take the collection $\{\phi_i\}_{i\in I_m}$ in $\mathbb{C}^n$ and suppose that $H_{n\times n}$ denotes the space of all $n \times n$ Hermitian matrices. Consider the operator $\Lambda_\Phi : H_{n\times n} \to \mathbb{R}^m$ defined by $\Lambda_\Phi(A) = \{\langle A, \phi_i \otimes \phi_i\rangle\}_{i\in I_m}$, in which $\phi_i \otimes \phi_i$ is the rank one projection onto $span\{\phi_i\}$. Then it is shown that $\Lambda_\Phi(f \otimes f) = \mathbb{M}_\Phi(f)$ and this equality makes a new identification of phase retrieval property. See [9, 14, 23] for more details.

Proposition 2.3. [23] $\mathbb{M}_\Phi$ is not injective if and only if there exists a matrix of rank 1 or 2 in the null space of Λ_Φ.

Moreover, the following equivalent conditions are easily obtained for a frame to do phase retrieval.

Corollary 2.4. [5] Let $\Phi = \{\phi_i\}_{i\in I_m}$ be a frame in $\mathcal{H}_n$, then the followings are equivalent;

(i) Φ does phase retrieval.

(ii) $\mathbb{M}_\Phi$ is injective.

(iii) $\Lambda_\Phi|_{B_1}$ is injective, where B_1 denotes rank one matrices.

(iv) There exists no rank 2 matrix in the null space of Λ_Φ.

A fundamental classification of frames which do phase retrieval was presented for finite dimensional real case in [16] and then for infinite dimensional case was presented in [22] in terms of the concept of complement property;

Definition 2.5. A family of vectors $\Phi = \{\phi_i\}_{i\in I}$ in $\mathcal{H}$ has the complement property if for every $\sigma \subset I$ either $\overline{span}\{\phi_i\}_{i\in\sigma} = \mathcal{H}$ or $\overline{span}\{\phi_i\}_{i\in\sigma^c} = \mathcal{H}$.

Theorem 2.6. [16] A family $\Phi = \{\phi_i\}_{i\in I}$ in a real Hilbert space $\mathcal{H}$ does phase retrieval if and only if it has the complement property.

It is worth noticing that the complement property is also necessary but not sufficient for injectivity of the mapping M_Φ in complex Hilbert spaces, see [23]. Although an extension of Theorem 2.6, which presents a necessary and sufficient condition for injectivity of the mapping M_Φ in $\mathbb{C}^n$, can be found in [46].

2.2 Phase retrieval dual frames

It is shown that the equivalent frames preserve phase retrieval property.

Theorem 2.7. [16] A family $\Phi = \{\phi_i\}_{i\in I}$ in $\mathcal{H}$ does phase retrieval if and only if $\{U\phi_i\}_{i\in I}$ does phase retrieval for every invertible operator U on $\mathcal{H}$.

Applying the above theorem, shows that a frame does phase retrieval if and only if its canonical dual does phase retrieval.

In [5] the authors used the above result and addressed the problem of characterizing phase retrieval dual frames of a given phase retrieval Φ in a finite dimensional Hilbert space. For some classes of frames it was shown that phase retrieval dual frames are dense in the set of all dual frames. Let Φ be a frame, D_Φ the set of all its dual frames and by and PD_Φ the subset of phase retrieval dual frames of Φ. The connection between phase retrieval duals of equivalent frames is given in the following lemma by a direct using of Theorem 2.7. The results in this subsection were appeared in [5].

Lemma 2.8. Suppose that $\Phi = \{\phi_i\}_{i\in I_m}$ is a frame for $\mathcal{H}_n$ and $\mathcal{T}$ is an invertible operator on $\mathcal{H}_n$. Then,

(i) $D_{\mathcal{T}\Phi} = (\mathcal{T}^*)^{-1} D_\Phi$.

(ii) $PD_{\mathcal{T}\Phi} = (\mathcal{T}^*)^{-1} PD_\Phi$.

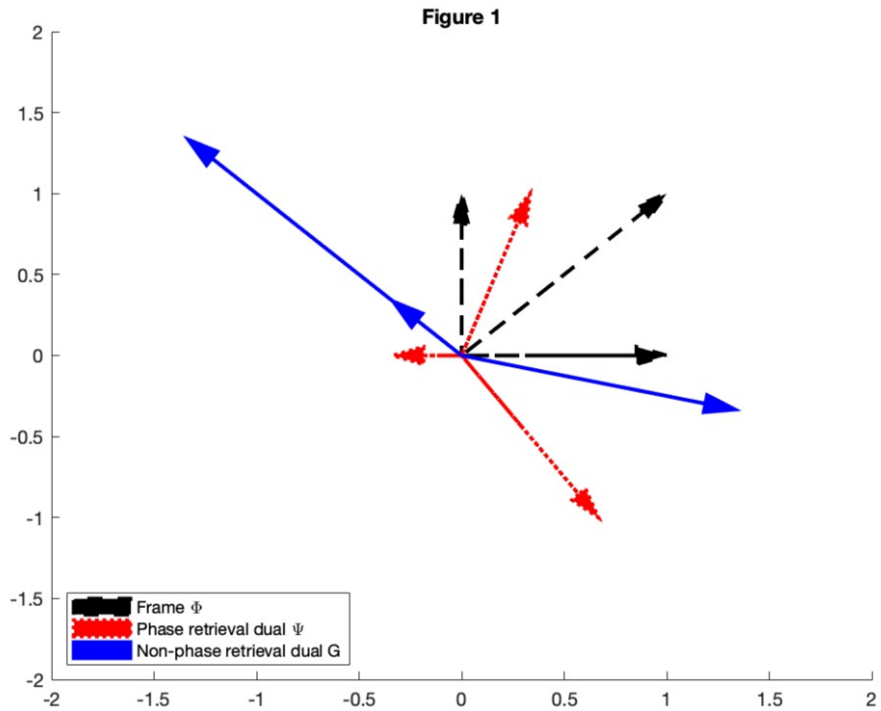
Theorem 2.9. Let $\phi = \{\phi_i\}_{i\in I_{2n-1}}$ be a phase retrieval frame in $\mathbb{R}^n$. Then PD_ϕ is an open and dense subset in D_ϕ.

The following example provides evidence that supports Theorem 2.9.

Example 2.10. Suppose that $\phi = \{\delta_1, \delta_2, \delta_1 + \delta_2\}$. In this case,

$$D_\phi = \left\{ \begin{bmatrix} \frac{2}{3} - x \\ \\ \frac{-1}{3} - y \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} - x \\ \\ \frac{2}{3} - y \end{bmatrix}, \begin{bmatrix} \frac{1}{3} + x \\ \\ \frac{1}{3} + y \end{bmatrix} ; x, y \in \mathbb{R} \right\}.$$

It is worthy of note that, all dual frames in this case are full spark and so phase retrieval except dual frames obtained by $(x,y) \in \mathbb{R}^2$ on three distinct lines $x = \frac{-1}{3}$, $y = \frac{-1}{3}$ and $y = \frac{1}{3} - x$. For instance, Put $x = 0$, $y = 2/3$ we get a phase retrieval dual Ψ and $x = 1, y = \frac{-2}{3}$, satisfying in $y = \frac{1}{3} - x$, gives a non-phase retrieval dual frame G. See Figure 1.



An analogous approach to the proof of Theorem 2.9 deduces that for any full spark frame Φ in $\mathbb{R}^n$ with $E(\Phi) = 1$, the set of all full spark dual frames is embedded into a generic set in $\mathbb{R}^n$.

Corollary 2.11. Let $\Phi = \{\phi_i\}_{i=1}^{n+1}$ be a full spark frame for $\mathbb{R}^n$. Then the set of all full spark dual frames of Φ is an open and dense subset of D_Φ and is embedded into a generic set in $\mathbb{R}^n$.

The following example shows that the condition that Φ is full spark in Corollary 2.11 is not necessary.

Example 2.12. Consider $\phi = \{\delta_1, \delta_2, \delta_3, \sum_{i=1}^{3} \delta_i, \delta_1 - \delta_2 + \delta_3\}$ as a (non-full spark) frame for $\mathbb{R}^3$. In this case, all dual frames are constructed by a generic choice of $\begin{bmatrix} u_4 \\ u_5 \end{bmatrix} \in \mathbb{R}^6$. In fact, by putting $u_4 = [x_1 \quad y_1 \quad z_1]^T$ and $u_5 = [x_2 \quad y_2 \quad z_2]^T$, the set of all dual frames of ϕ, are given by

$$g_1 = \begin{bmatrix} \frac{3}{5} - x_1 - x_2 \\ \\ -y_1 - y_2 \\ \\ \frac{-2}{5} - z_1 - z_2 \end{bmatrix}, g_2 = \begin{bmatrix} x_2 - x_1 \\ \\ \frac{1}{3} + y_2 - y_1 \\ \\ -z_2 - z_1 \end{bmatrix},$$

$$g_3 = \begin{bmatrix} \frac{-2}{5} - x_1 - x_2 \\ \\ -y_1 - y_2 \\ \\ \frac{3}{5} - z_1 - z_2 \end{bmatrix}, g_4 = \begin{bmatrix} \frac{1}{5} + x_1 \\ \\ \frac{1}{3} + y_1 \\ \\ \frac{1}{5} + z_1 \end{bmatrix}, g_5 = \begin{bmatrix} \frac{1}{5} + x_1 \\ \\ \frac{-1}{3} + y_2 \\ \\ \frac{1}{5} + z_2 \end{bmatrix}$$

where $x_1, x_2, y_1, y_2, z_1, z_2$ are obtained arbitrarily from $\mathbb{R}$. And all cases in which a dual frame fails to do phase retrieval is associated to the roots of a 6-variable polynomial in $\mathbb{R}^6$ given by multiplying of the polynomials as $det[g_{i_1}|g_{i_2}|g_{i_3}] = 0$, for all index set $\{i_j\}_{j=1}^{3} \subset I_5$. As one case, $det[g_2|g_4|g_5] = 0$ deduces that

$$z_1 + x_2 - \frac{3x_1}{5} - \frac{3z_2}{5} + 3x_2z_1 - 3x_1z_2 = 0,$$

and the roots of this equation, which are associated to a family of non-phase retrieval dual frames, constitute a surface in $\mathbb{R}^3$.

2.3 Topological point of view

Recently a new topological approach has been presented in [7] for identifying phase retrieval frames. The motivation of this method comes back to the topological structure of the projective Hilbert spaces. Indeed, a projective Hilbert space is a quotient space associated to a non-zero Hilbert space containing the equivalent classes by identifying two non-zero vectors which differ by a complex factor, see [15, 18, 43, 44]. Next, for a family of vectors Φ in a Hilbert space $\mathcal{H}$ the authors have introduced an initial topology τ_Φ on the quotient space $\hat{\mathcal{H}}$ and establish a topology-based method on the quotient space. This method presents a new classification of phase retrieval property in both finite and infinite dimensional cases.

To describe the method briefly, consider the sequence $\Phi = \{\phi_i\}_{i\in I}$ in Hilbert space $\mathcal{H}$. Then the collection $\{\rho_{\phi_i}\}_{i\in I}$ induces an initial topology τ_Φ on $\hat{\mathcal{H}}$, where $\rho_{\phi_i} : \hat{\mathcal{H}} \to \mathbb{R}$ is defined by

$$\rho_{\phi_i}\hat{x} = |\langle x, \phi_i\rangle|.$$

This topology is the coarsest topology on $\hat{\mathcal{H}}$ that makes the function family $\{\rho_{\phi_i}\}_{i\in I}$ continuous with the subbases $\{\rho_{\phi_i}^{-1}(r_1, r_2) : r_1, r_2 \in \mathbb{R}^+, i \in I\}$. Studying the structure of τ_Φ is the main aim of this section. We first note that in the case $\Phi = \{\phi_i\}_{i\in I}$ does phase retrieval in $\mathcal{H}$, the space $(\hat{\mathcal{H}}, \tau_\Phi)$ is T_1. Indeed, it is sufficient to show that $\{\hat{x}\}^c$ is open for every $\hat{x} \in \hat{\mathcal{H}}$. Consider $\hat{y} \in \{\hat{x}\}^c$ then $\hat{x} \neq \hat{y}$ and so there exists $i \in I$ such that $|\langle x, \phi_i\rangle| \neq |\langle y, \phi_i\rangle|$, we may assume that

$$|\langle y, \phi_i\rangle| < |\langle x, \phi_i\rangle|.$$

Hence, $\hat{y} \in \rho_{\phi_i}^{-1}(0, |\langle x, \phi_i\rangle|) \subseteq \{\hat{x}\}^c$, i.e., $\{\hat{x}\}^c$ is an open set in $\hat{\mathcal{H}}$.

As we mentioned, a fundamental classification of phase retrieval property infinite dimensional real Hilbert spaces in based on the complement property. The following proposition presents a new characterization of the phase retrieval in a general case.

Proposition 2.13. [7] $\Phi = \{\phi_i\}_{i\in I}$ does phase retrieval in $\mathcal{H}$ if and only if $(\hat{\mathcal{H}}, \tau_\Phi)$ is Hausdorff.

Furthermore, the following theorem provides a characterization of phase retrieval property in finite dimensional case.

Theorem 2.14. [7] Let $\Phi = \{\phi_i\}_{i\in I_m}$ be a frame for the finite dimensional Hilbert space $\mathcal{H}_n$. Then Φ does phase retrieval in $\mathcal{H}_n$ if and only if τ_Φ and τ_w are equivalent.

In particular, it is proved that τ_Φ coincides with weak topology τ_w on $\hat{\mathcal{H}}$ in the finite dimensional case if and only if Φ does phase retrieval, where by τ_w we mean the coarsest topology on $\hat{\mathcal{H}}$ that makes the function family $\{\rho_y\}_{y\in\mathcal{H}}$, defined as

$$\rho_y(\hat{x}) = |\langle x, y\rangle|, \quad (\hat{x} \in \hat{\mathcal{H}})$$

continuous. For more discussions on metric and topological structure of the quotient space $\hat{\mathcal{H}}$ we refer to [7, 21, 31, 8, 18, 44].

2.4 Infinite dimensional case

Phase retrieval sequences in infinite dimensional Hilbert spaces were first studied in [22]. In this case the complement property is defined similarly and it is shown that if a frame in a separable Hilbert space does phase retrieval then it has complement property. Moreover, on a separable real Hilbert space, a frame has the complement property if and only if it does phase retrieval. See [22]. However, there are some substantial differences between phase retrieval sequences in finite and infinite dimensional cases. For instance, unlike finite-dimensional setting, a frame can be perturbed by a small amount to arrive at a frame that does not do phase retrieval. That means phase retrieval in infinite-dimensional Hilbert spaces is not stable [22].

The other approach for identifying phase retrieval frames is presented in terms of the Bi-Lipschitz property of the non-linear mapping $\mathbb{M}_\Phi$ with respect to appropriate metrics on the quotient space in both real and complex Hilbert spaces. See [12] for more details and information. To this end the authors considered a natural choice of metric on the quotient space so called the Bures-Wasserstein metric defined by

$$D(\hat{x}, \hat{y}) = \min_{\theta} \|x - e^{i\theta} y\| = \sqrt{\|x\|^2 + \|y\|^2 - 2|\langle x, y\rangle|}. \tag{2.3}$$

Then, it was shown that if Φ is phase retrieval the mapping $\mathbb{M}_\Phi$ has bi-Lipschitz bounds with respect this metric. Although in infinite setting Cahill et.al proved the following result;

Theorem 2.15. Let $\Phi = \{\phi_i\}_{i\in I}$ be a frame for an infinite-dimensional separable Hilbert space $\mathcal{H}$. Moreover, let $\|\phi_i\| \geq c > 0$ for all $i \in I$. Then, for every $\delta > 0$, there exist $f, g \in \mathcal{H}$ so that $D(\hat{f}, \hat{g}) \geq 1$ but $\|\mathbb{M}_\Phi f - \mathbb{M}_\Phi g\| < \delta$.

To overcome this unstability, Calderbank et. al [24], considered a generalization of phase retrieval to the setting of subspaces of $L^2(\mathbb{R})$ and proved that there exist infinite dimensional subspaces of $L^2(\mathbb{R})$ where phase retrieval is stable. This method also suggests a new approach for uniform stability of phase retrieval property in finite dimensional case. See also [29] for an extension of this method on $L^p(\mathbb{R})$, $1 \leq p \leq \infty$.

In [20] also Casazza et .al investigated the similarities and differences of phase retrieval property in infinite dimensional with respect to finite setting. They showed that some properties are preserved in infinite case, however; the results confirmed that there are essential differences. As we know in finite dimensional doing phase retrieval is equivalent to complement property which immediately deduces that phase retrieval sequences are frames in Hilbert spaces. However, the following example shows that the same does not hold in infinite dimensional case.

Example 2.16. [20] Let $\Phi = \{e_i + e_j\}_{i<j}$, where $\{e_i\}_{i=1}^{\infty}$ is the canonical orthonormal basis of l^2. Then Φ does phase retrieval in l^2, although Φ is not a frame in l^2.

Also, it was shown that, however as mentioned in [22] that phase retrieval property is unstable in infinite case, it is possible to perturb a family doing phase retrieval in l^2 as long as we perturb all vectors in the same direction. See [20] for more details and properties of phase retrieval in l^2.

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