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Some inequalities related to triangle inequality

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Abstract

We deal with three well-known inequalities in normed spaces concerning the triangle inequality. The first is about (p,q)-angular distance which is a generalization of the notion of angular distance and p-angular distance discussed in the literature. The second relates to the Cauchy-Schowarz inequality. Finally, we introduce a norm in an inner product space through a given linear operator T and then we state a refinement of triangle inequality via the original norm of the space.

Keywords: Dunkle–Williams inequality, Cauchy-Schowarz inequality, Triangle inequality, (p,q)-angular distance.
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1 Introduction

In this paper, we try to generalize some well-known inequalities. The first, relates to the well-known Dunkle–Williams inequality concerning the notion of angular distance. The angular distance of two nonzero vectors x and y in a normed space $\mathscr{X}$ is given by

$$\alpha[x,y]=\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|.$$

In 1964 Dunkl and Williams proved the following inequality in which they obtained an upper bound for angular distance in a normed space $\mathscr{X}$

$$\alpha\leq\frac{4\|x-y\|}{\|x\|+\|y\|},\qquad(x,y\in\mathscr{X}-\{0\}).$$

They also proved that in an inner product space the constant 4 could be replaced by 2. Surprisingly, this case happens only in inner product spaces. Thus it characterizes inner product spaces. Kirk and Smily showed this last fact, see [2]. In [3] Maligranda regarded the following notion of p-angular distance as a generalization of angular distance for which he proved some inequalities corresponding to different states of the parameter p in $\mathbb{R}$;

$$\alpha_p[x,y]=\left\|\frac{x}{\|x\|^{1-p}}-\frac{y}{\|y\|^{1-p}}\right\|,\qquad(x,y\in\mathscr{X}-\{0\}).$$

In [1] the authors state a characterization of inner product spaces through an inequality for p-angular distance. One could regard the following (p,q)-angular distance in normed space $\mathscr{X}$ as a generalization of p-angular distance

$$\alpha_{(p,q)}[x,y]=\left\|\frac{x}{\|x\|^{1-p}}-\frac{y}{\|y\|^{1-q}}\right\|,\qquad(x,y\in\mathscr{X}).$$

Note that the definition reduces to the definition of p-angular distance whenever $p=q$.

Then, we deal with the famous Cuachy–Showarz inequality in inner product spaces. The Cauchy-Schwarz inequality states that for two vectors x and y of an inner product space $(\mathscr{X},\langle\cdot,\cdot\rangle)$ we have that $|\langle x,y\rangle|\leq\|x\|\|y\|$. There is enough evidence to justify the wide application of this inequality in mathematics. It is considered to be one of the most important inequalities in mathematics. Here in this note, we are going to generalize this inequality.

Let $x,x_1,\cdots,x_n$ be $n+1$ vectors in $\mathscr{X}$. Invoking the Cauchy–Schwarz inequality n times, we come to

$$\sum_{i=1}^{n}|\langle x,x_i\rangle|\leq\|x\|\sum_{i=1}^{n}\|x_i\|.$$

In this note, we want to state a refinement of this last inequality.

Finally, we state an inequality in inner product spaces through operators defined on such spaces.

2 (p,q)-angular distance

We start this section with the following inequalities.

Lemma 2.1. Let $x, y \in \mathscr{X}$ and $p \in \mathbb{R}$. Then

$$\|x\| |\|x\|^{p-1} - \|y\|^{p-1}| \leq \begin{cases} (1-p)\frac{|\|y\|-\|x\||}{\|y\|^{1-p}} & p<1 \\ p\frac{|\|y\|-\|x\||}{\|y\|^{1-p}} & p>1. \end{cases} \tag{2.1}$$

Proof . We only check the case when $\|x\| > \|y\|$ and $p < 1$. The others are verified similarly. We follow the method of [3, Theorem 2]. For $b \geq a \geq 0$, we have that

$$b^{1-p} - a^{1-p} = (1-p)\int_a^b t^{-p}\mathrm{d}t \leq (1-p)a^{-p}(b-a). \tag{2.2}$$

Therefore,

$$\begin{aligned} \|x\|(\|x\|^{p-1} - \|y\|^{p-1}) &= \|x\|\left(\frac{\|y\|^{1-p} - \|x\|^{1-p}}{\|y\|^{1-p}\|x\|^{1-p}}\right) \\ &\leq (1-p)\frac{|\|y\| - \|x\||}{\|y\|\|x\|^{-p}} \qquad \text{(by (2))} \\ &\leq (1-p)\frac{|\|y\| - \|x\||}{\|y\|^{1-p}}. \end{aligned}$$

□

Theorem 2.2. Let $x, y \in \mathscr{X}$ and let $p, q \in \mathbb{R}$. Then

$$\alpha_{(p,q)}[x,y] \leq \min\left\{(2-p)\frac{\|x-y\|}{\|y\|^{1-p}} + |\|y\|^q - \|y\|^p|,\ (2-q)\frac{\|x-y\|}{\|x\|^{1-q}} + |\|x\|^q - \|x\|^p|\right\}$$

whenever $p, q \leq 1$ and

$$\alpha_{(p,q)}[x,y] \leq \min\left\{p\frac{\|x-y\|}{\|y\|^{1-p}} + |\|y\|^q - \|y\|^p|,\ q\frac{\|x-y\|}{\|x\|^{1-q}} + |\|x\|^q - \|x\|^p|\right\}$$

whenever $1 < p, q$, and

$$\alpha_{(p,q)}[x,y] \leq \min\left\{p\frac{\|x-y\|}{\|y\|^{1-p}} + |\|y\|^q - \|y\|^p|,\ (2-q)\frac{\|x-y\|}{\|x\|^{1-q}} + |\|x\|^q - \|x\|^p|\right\}$$

whenever $q \leq 1 < p$.

Proof . First, we note that

$$\begin{aligned}\alpha_{p,q}[x,y] &= \|\|x\|^{p-1}x-\|y\|^{q-1}y\| \\ &\leq \|\|x\|^{p-1}x-\|y\|^{p-1}x\|+\|\|y\|^{p-1}x-\|y\|^{p-1}y\|+\|\|y\|^{p-1}y-\|y\|^{q-1}y\| \\ &= \|x\|\|\|x\|^{p-1}-\|y\|^{p-1}|+\|y\|^{p-1}\|x-y\|+|\|y\|^{p}-\|y\|^{q}|.\end{aligned}$$

and analogously we have that

$$\alpha_{p,q}[x,y] \leq \|y\|\|\|y\|^{q-1}-\|x\|^{q-1}|+\|x\|^{q-1}\|x-y\|+|\|x\|^{p}-\|x\|^{q}|.$$

Now, we apply inequalities (2.1) of Lemma 2.1 to these inequalities, and we are done. □

Remark 2.3. In [4] the authors considered a type of orthogonality concerning p-angular distance. One could generalize the results of [4]. The notion of p-angular distance orthogonality is generalized routinely as follows;

for $p,q\in\mathbb{R}$ and for given x,y in normed space $\mathscr{X}$, vector x is said to be (p,q)-angular distance orthogonal to vector y ($x\perp_A^{(p,q)} y$) whenever

$$\|x\|\|y\|=0 \text{ or } \alpha_{(p,q)}[x,y]=\alpha_{(p,q)}[x,-y].$$

3 Cauchy-Schwarz inequality

Before stating our result in this section let us fix a notation that is needed in it. By $\langle x,y,z\rangle$ we mean

$$\mathrm{Sgn}(\langle x,y\rangle)\mathrm{Sgn}(\langle z,x\rangle)\langle y,z\rangle$$

where Sgn stands for the sign function defined by

$$\mathrm{Sgn}(z):=\begin{cases}\frac{z}{|z|} & z\neq 0\\ 0 & z=0\end{cases}.$$

Obviously $\langle x,y,z\rangle\leq\|y\|\|z\|$.

Theorem 3.1. Let $(\mathscr{X},\langle\cdot,\cdot\rangle)$ be an inner product space, $n\in\mathbb{N}$ with $n>1$, and let $x,x_1,\cdots,x_n$ be $n+1$ vectors in $\mathscr{X}$. Then

$$\sum_{i=1}^{n}|\langle x,x_i\rangle|\leq\|x\|\sqrt{\sum_{i=1}^{n}\|x_i\|^2+\sum_{i,j}\langle x,x_i,x_j\rangle}.$$

If the equality holds, then x is a linear combination of $x_1,\cdots,x_n$.

Proof . Let $\alpha_1, \cdots, \alpha_n$ be n complex numbers. Then

$$\begin{aligned} \|x+\alpha_1 x_1+\cdots+\alpha_n x_n\|^2 &= \|x\|^2+\sum_{i=1}^{n}|\alpha_i|^2\|x_i\|^2+\sum_{i=1}^{n}\overline{\alpha_i}\langle x, x_i\rangle \\ &\quad +\sum_{i=1}^{n}\alpha_i\langle x_i, x\rangle+\sum_{i,j=1}^{n}\alpha_i\overline{\alpha_j}\langle x_i, x_j\rangle. \end{aligned}$$

For $t \in \mathbb{R}$ assume that $\alpha_i = t\mathrm{Sgn}(\langle x, x_i\rangle)$. Without loss of generality, we may assume that $\alpha_i \neq 0$. Thus we have that

$$\begin{aligned} 0 \leq \|x+\alpha_1 x_1+\cdots+\alpha_n x_n\|^2 &= \|x\|^2+2t\sum_{i=1}^{n}|\langle x, x_i\rangle| \\ &\quad +t^2\left(\sum_{i=1}^{n}\|x_i\|^2+\sum_{i,j=1}^{n}\langle x, x_i, x_j\rangle\right). \end{aligned} \tag{3.1}$$

The expression on the left is quadratic with respect to parameter t and is positive for every $t \in \mathbb{R}$. Therefore

$$\sum_{i=1}^{n}\|x_i\|^2+\sum_{i,j}\langle x, x_i, x_j\rangle \geq 0$$

and

$$\Delta := \left(2\sum_{i=1}^{n}|\langle x, x_i\rangle|\right)^2-4\|x\|^2\left(\sum_{i=1}^{n}\|x_i\|^2+\sum_{i,j=1}^{n}\langle x, x_i, x_j\rangle\right) \leq 0$$

which implies that

$$\sum_{i=1}^{n}|\langle x, x_i\rangle| \leq \|x\|\sqrt{\sum_{i=1}^{n}\|x_i\|^2+\sum_{i,j}\langle x, x_i, x_j\rangle}.$$

Now, we note that the equality takes place in the inequality in relations (3.2) if $\Delta = 0$, and we are done. □

Remark 3.2. From the proof of this theorem we see that

$$\sum_{i=1}^{n}\|x_i\|^2+\sum_{i,j}\langle x, x_i, x_j\rangle \geq 0.$$

Thus

$$\sum_{i,j}\langle x, x_i, x_j\rangle \in \mathbb{R}$$

for any choice of vectors $x, x_1, \cdots, x_n$.

Now we generalize Theorem 3.2 considering a continuous version of it.

Theorem 3.3. Let (X,μ) be a measure space, $f : X\times X \to \mathbb{C}$ and $g : X \to \mathbb{C}$ integrable functions. Then

$$\begin{aligned}&\left|\int_X\int_X g(x)\overline{f(x,y)}d\mu_x\right| d\mu_y \leq \\ &\left(\int_X g(x)\overline{g(x)}d\mu_x\right)\\ &\times\left(\int_X\int_X\int_X \operatorname{Sgn}\left(\int_X g(x)\overline{f(x,z)}\right) f(x,z)\overline{\operatorname{Sgn}}\left(\int_X g(x)\overline{f(x,y)}\right)\overline{f(x,y)}d\mu_x d\mu_y d\mu_z\right).\end{aligned}$$

Proof . Let $h : X \to \mathbb{C}$ be a measurable function. Then

$$\begin{aligned}0 &\leq \left(\int_X (g(x)+\int_X h(y)f(x,y)d\mu_y\right)\left(\overline{g(x)+\int_X h(y)f(x,y)d\mu_y)}d\mu_x\right)\\ &= \int_X g(x)\overline{g(x)}d\mu_x + \int_X\int_X h(y)\overline{g(x)}f(x,y)d\mu_x d\mu_y\\ &\quad+\int_X\int_X g(x)\overline{h(y)f(x,y)}d\mu_x d\mu_y + \int_X\int_X\int_X h(z)\overline{h(y)}f(x,z)\overline{f(x,y)}d\mu_x d\mu_y d\mu_z.\end{aligned}$$

For $t\in\mathbb{R}$ assume that $h(y) = t\operatorname{Sgn}(\int_X g(x)\overline{f(x,y)}d\mu_x)$. Thus we have that

$$\begin{aligned}&\left(\int_X (g(x)+\int_X h(y)f(x,y)d\mu_y\right)\left(\overline{g(x)+\int_X h(y)f(x,y)d\mu_y)}d\mu_x\right)\\ &= \int_X g(x)\overline{g(x)}d\mu_x + 2t\int_X\left|\int_X \overline{g(x)}f(x,y)d\mu_x\right|d\mu_y\\ &\quad+t^2\int_X\int_X\int_X \operatorname{Sgn}\left(\int_X g(x)\overline{f(x,z)}\right)f(x,z)\overline{\operatorname{Sgn}}\left(\int_X g(x)\overline{f(x,y)}\right)\overline{f(x,y)}d\mu_x d\mu_y d\mu_z.\end{aligned}$$

The expression on the left is quadratic with respect to parameter t and is positive for every $t\in\mathbb{R}$. Therefore

$$\int_X\int_X\int_X \operatorname{Sgn}\left(\int_X g(x)\overline{f(x,z)}\right)f(x,z)\overline{\operatorname{Sgn}}\left(\int_X g(x)\overline{f(x,y)}\right)\overline{f(x,y)}d\mu_x d\mu_y d\mu_z \geq 0$$

and

$$\begin{aligned}\Delta &:= \left(2\int_X\left|\int_X \overline{g(x)}f(x,y)d\mu_x\right|d\mu_y\right)^2\\ &\quad -4\left(\int_X g(x)\overline{g(x)}d\mu_x\right)\\ &\times\left(\int_X\int_X\int_X \operatorname{Sgn}\left(\int_X g(x)\overline{f(x,z)}\right)f(x,z)\overline{\operatorname{Sgn}}\left(\int_X g(x)\overline{f(x,y)}\right)\overline{f(x,y)}d\mu_x d\mu_y d\mu_z\right)\leq 0\end{aligned}$$

and we are done. □

4 A refinement of the triangle inequality

Let T be an operator on inner product space $(\mathscr{X}, \langle\cdot,\cdot\rangle)$ and define

$$\|x\|_1^2 := \|Tx\|^2 + \|x\|^2.$$

It is readily verified that $\|\cdot\|_1$ defines really a norm on $\mathscr{X}$. By a closer look at norm $\|\cdot\|_1$ we could state a refinement for the triangle inequality utilizing the original norm on $\mathscr{X}$ as follows.

Theorem 4.1. For any $x, y \in \mathscr{X}$ we have that

$$\begin{aligned}\|x+y\|_1 &\leq \\ \sqrt{\|T(x)\|^2 + \|T(y)\|^2 + 2\|T(x)\|\|T(y)\| + \|y\|^2 + \|x\|^2 + 2\|x\|\|y\|} &\leq \\ \|x\|_1 + \|y\|_1.\end{aligned}$$

Proof . Given two vectors x and y the triangle inequality for $\|\cdot\|_1$ is as follows

$$\sqrt{\|T(x+y)\|^2 + \|x+y\|^2} \leq \sqrt{\|T(x)\|^2 + \|x\|^2} + \sqrt{\|T(y)\|^2 + \|y\|^2}. \tag{4.1}$$

On the other hand by the triangle inequality for $\|\cdot\|$ we have that

$$\begin{aligned}\sqrt{\|T(x+y)\|^2 + \|x+y\|^2} &\leq \\ \sqrt{\|T(x)\|^2 + \|T(y)\|^2 + 2\|T(x)\|\|T(y)\| + \|y\|^2 + \|x\|^2 + 2\|x\|\|y\|}.\end{aligned} \tag{4.2}$$

Now it is not difficult to see that

$$\begin{aligned}\sqrt{\|T(x)\|^2 + \|T(y)\|^2 + 2\|T(x)\|\|T(y)\| + \|y\|^2 + \|x\|^2 + 2\|x\|\|y\|} &\leq \\ \sqrt{\|T(x)\|^2 + \|x\|^2} + \sqrt{\|T(y)\|^2 + \|y\|^2}.\end{aligned} \tag{4.3}$$

Thus (4.1),(4.2) and (4.3) bring us to

$$\begin{aligned}\|x+y\|_1 &\leq \\ \sqrt{\|T(x)\|^2 + \|T(y)\|^2 + 2\|T(x)\|\|T(y)\| + \|y\|^2 + \|x\|^2 + 2\|x\|\|y\|} &\leq \\ \|x\|_1 + \|y\|_1\end{aligned}$$

which indicates a refinement of the triangle inequality of norm $\|\cdot\|_1$.$\square$

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