



Generalized frame generators for unitary systems on Hilbert spaces

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Abstract

In this paper, we study the concept of generalized Bessel (frames, Riesz basis and orthonormal basis) generators for a unitary system of operators on a separable Hilbert space. Some characterizations of these generators are presented in term of a generalized wandering generator for the unitary system. Additionally, we prove that the set of wandering generator for a unitary system is closed under some combinations.

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1 Introduction and Preliminaries

Frames in Hilbert spaces were first introduced by Duffin and Schaeffer to study nonharmonic Fourier series in 1952 [2], and were widely considered from 1986, since the great work

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by Daubechies et al. [4]. Now frames play an important role not only in the theory but also in many applications and have been widely applied in signal processing [8], sampling [6, 7], coding and communications [17], filter bank theory [3], system modeling [5], and so on. In recent decades, many generalizations of frames were studied by authors. One of them, and the most applicable generalization of frames namely frame generators for unitary systems of operators on Hilbert spaces studied by Han and Larson [13]. Frames of translates and Gabor frames are special case of these frames. In this paper we are going to introduce an operator generalization of frame generator for unitary systems. For this, we need some preliminaries.

Let H and K be separable Hilbert spaces and $B(H, K)$ be the set of all bounded linear operators from H into K . We $B(H)$ for $B(H, H)$. A sequence $\{f_n\}_{n \in \mathbb{N}}$ in H is called a *frame* if it satisfies the following inequalities

$$a\|f\|^2 \leq \sum_{n \in \mathbb{N}} |\langle f, f_n \rangle|^2 \leq b\|f\|^2, \quad (f \in H),$$

for some $a, b > 0$. It is called a tight frame if $a=b$ and a Parseval frame if $a = b = 1$.

A *unitary system* \mathcal{U} is a subset of the unitary operators on H which contains the identity operator I . A unitary group is a special case of unitary system when \mathcal{U} is a group. An element $\psi \in H$ is called a *wandering vector* for \mathcal{U} if $\mathcal{U}\psi := \{U\psi : U \in \mathcal{U}\}$ is an orthonormal set. If $\mathcal{U}\psi$ is an orthonormal basis for H , then ψ is called a *complete wandering vector* for \mathcal{U} . The set of all complete wandering vectors for \mathcal{U} is denoted by $\mathcal{W}(\mathcal{U})$.

Analogously, a *Bessel generator* for \mathcal{U} is a vector $\eta \in H$ with the property that $\mathcal{U}\eta$ is a Bessel sequence for H . The set of all Bessel generators for \mathcal{U} is denoted by $\mathcal{B}_{\mathcal{U}}$. A *complete frame generator* for \mathcal{U} is a vector $\eta \in H$ with the property that $\mathcal{U}\eta$ is a frame for H . The set of all complete frame generators for \mathcal{U} is denoted by $\mathcal{F}_{\mathcal{U}}$. For more information on frame generators for unitary systems see references [1, 9, 11, 12, 13, 15].

For a unitary system \mathcal{U} and a vector $\psi \in H$, the *local commutant* $C_{\psi}(\mathcal{U})$ at ψ is the set of all bounded linear operators T on H for which $(TU - UT)\psi = 0$, for all $U \in \mathcal{U}$. Clearly $C_{\psi}(\mathcal{U})$ contains the commutant \mathcal{U}' of \mathcal{U} . It can be seen that if \mathcal{U} is a unitary group, then $C_{\psi}(\mathcal{U})$ is indeed the commutant of \mathcal{U} .

The following are two important propositions for unitary systems on separable Hilbert spaces which will be used in the rest of the paper. For the proofs we refer to Proposition 1.3 and Proposition 1.5 [13].

Proposition 1.1. Suppose that ψ is a complete wandering vector for a unitary system \mathcal{U} . Then

- i) a vector η is a normalized tight frame vector for \mathcal{U} if and only if there is a (unique) partial isometry $A \in C_{\psi}(\mathcal{U})$ such that $A\psi = \eta$.

- ii) a vector η is a complete normalized tight frame vector for \mathcal{U} if and only if there is a (unique) co-isometry $A \in C_\psi(\mathcal{U})$ such that $A\psi = \eta$.

Proposition 1.2. Let \mathcal{U} and ψ be as in Proposition 1.1. Then a vector η is a frame vector with frame bounds a and b if and only if there exists an (unique) operator $A \in C_\psi(\mathcal{U})$ such that $\eta = A\psi$ and $aP \leq AA^* \leq bP$ for some orthogonal projection P . In fact P is the orthogonal projection onto $\overline{\text{span}}(\mathcal{U}\eta)$.

We also need the concept of a generalized frames, which introduced by Sun [18]. Suppose that H and $\{K_i\}_{i \in \mathbb{N}}$ are Hilbert spaces. A sequence $\{\Lambda_i\}_{i \in \mathbb{N}} \subset B(H, K_i)$ is called a *generalized frame*, or simply a *g-frame* for H with respect to $\{K_i\}_{i \in \mathbb{N}}$ if there exist two positive constants a and b such that

$$a\|f\|^2 \leq \sum_{i \in \mathbb{N}} \|\Lambda_i f\|^2 \leq b\|f\|^2, \quad (f \in H).$$

The numbers a and b are called the lower and upper frame bounds, respectively. In the case that $K_i = K$ we say $\{\Lambda_i\}_{i \in \mathbb{N}}$ is *g-frame* for H with respect to K . As usual, $\{\Lambda_i\}_{i \in \mathbb{N}}$ is called a tight *g-frame* if $a = b$ and a Parseval *g-frame* if $a = b = 1$. We say $\{\Lambda_i\}_{i \in \mathbb{N}}$ is a *g-frame* sequence, if it is a *g-frame* for $\overline{\text{span}}\{\Lambda_i^*(K_i)\}_{i \in \mathbb{N}}$. If we have only the second inequality, then it is called a *g-Bessel* sequence. The sequence $\{\Lambda_i\}_{i \in \mathbb{N}}$ is called linear independent, if $\{\Lambda_i^* g_i\}_{i \in \mathbb{N}}$ is linear independent for any $g_i \in K_i$.

We say $\{\Lambda_i\}_{i \in \mathbb{N}}$ is a *g-Riesz basis* if the only $f \in H$ which satisfies $\Lambda_i f = 0$, for all $i \in \mathbb{N}$ is zero and there are two positive constants a and b such that

$$a \sum_{i \in \mathbb{J}} \|g_i\|^2 \leq \left\| \sum_{i \in \mathbb{J}} \Lambda_i^* g_i \right\|^2 \leq b \sum_{i \in \mathbb{J}} \|g_i\|^2.$$

for any finite subset $\mathbb{J} \subset \mathbb{N}$ and $g_i \in K_i, i \in \mathbb{J}$.

A sequence $\{\Lambda_i\}_{i \in \mathbb{N}}$ of operators is said to be a *g-orthonormal basis* for H if for every $f \in H$, $i, j \in \mathbb{N}$, $g_i \in K_i$ and $g_j \in K_j$, it satisfies

$$\sum_{i \in \mathbb{N}} \|\Lambda_i f\|^2 = \|f\|^2, \quad \langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle = \delta_{i,j} \langle g_i, g_j \rangle.$$

It is well-known that the space

$$\left(\sum_{i \in \mathbb{N}} \oplus K_i \right)_{l^2} := \left\{ \{g_i\}_{i \in \mathbb{N}} : g_i \in K_i, i \in \mathbb{N}, \sum_{i \in \mathbb{N}} \|g_i\|^2 < +\infty \right\}$$

with the inner product defined by $\langle \{f_i\}, \{g_i\} \rangle = \sum_{i \in \mathbb{N}} \langle f_i, g_i \rangle$ is a Hilbert space with point-wise operations. Let $\Lambda = \{\Lambda_i\}_{i \in \mathbb{N}}$ be a *g-Bessel* sequence. The synthesis operator of $\{\Lambda_i\}_{i \in \mathbb{N}}$ is defined as follows,

$$U_\Lambda : \left(\sum_{i \in \mathbb{N}} \oplus K_i \right)_{l^2} \rightarrow H; \quad U_\Lambda(\{g_i\}_{i \in \mathbb{N}}) = \sum_{i \in \mathbb{J}} \Lambda_i^* g_i.$$

The analysis operator is the adjoint operator of U_Λ which is defined by

$$U_\Lambda^* : H \rightarrow \left(\sum_{i \in \mathbb{N}} \oplus K_i \right)_{l^2}; \quad U_\Lambda^*(f) = \{\Lambda_i f\}_{i \in \mathbb{N}}.$$

By using the synthesis operator and the adjoint operator, the g -frame operator S_Λ is defined by

$$S_\Lambda f = U_\Lambda U_\Lambda^* f = \sum_{i \in \mathbb{N}} \Lambda_i^* \Lambda_i f, \quad (f \in H),$$

which is a bounded, positive and invertible operator. So the following reconstruction formula holds for all $f \in H$

$$f = S_\Lambda S_\Lambda^{-1} f = S_\Lambda^{-1} S_\Lambda f = \sum_{i \in \mathbb{N}} \Lambda_i^* \Lambda_i S_\Lambda^{-1} f = \sum_{i \in \mathbb{N}} S_\Lambda^{-1} \Lambda_i^* \Lambda_i f.$$

The sequence $\{\Lambda_i S_\Lambda^{-1}\}_{i \in \mathbb{N}}$ is called the canonical dual g -frame of $\{\Lambda_i\}_{i \in \mathbb{N}}$.

In this paper, we are going to define a generalized form of Bessel (frame, Riesz basis, orthonormal basis) generator vectors for a unitary system on a separable Hilbert space. Based on the standard results of generators for unitary systems, we provide some characterizations of Bessel (normalized tight frame, frame, Riesz basis and orthonormal basis) generators for a unitary system in terms of a generalized wandering generator for the unitary system. Additionally, we will examine the set of generalized wandering generator for a unitary system. It will be proved that this set is closed under some combinations.

2 Main result

Suppose H and K are two separable Hilbert spaces, $\Lambda \in B(H, K)$ and \mathcal{U} is a unitary system on H . We denote by $\Lambda_{\mathcal{U}}$ the set $\Lambda_{\mathcal{U}} := \{\Lambda \circ U : U \in \mathcal{U}\}$. The local commutant of \mathcal{U} in Λ is defined by

$$C_\Lambda(\mathcal{U}) = \{T \in B(H) : \Lambda T^* U = \Lambda U T^*, \quad U \in \mathcal{U}\}.$$

Definition 2.1. The operator $\Lambda \in B(H, K)$ is called g -Bessel (respectively, *complete g -frame*, *complete g -Riesz*, *complete g -wandering*) generator for \mathcal{U} or \mathcal{U} - g -Bessel (respectively, *complete \mathcal{U} - g -frame*, *complete \mathcal{U} - g -Riesz*, *complete \mathcal{U} - g -wandering*) if $\Lambda_{\mathcal{U}}$ is g -Bessel (respectively, complete g -frame, complete g -Riesz, complete g -wandering) for H with respect to K . We denote by $g\text{-}\mathcal{B}_{\mathcal{U}}(H; K)$ (respectively, $g\text{-}\mathcal{CF}_{\mathcal{U}}(H; K)$, $g\text{-}\mathcal{CR}_{\mathcal{U}}(H; K)$, $g\text{-}\mathcal{CW}_{\mathcal{U}}(H; K)$) the set of all \mathcal{U} - g -Bessel (respectively, complete g -frame, complete g -Riesz, complete g -wandering) generator for H with respect to K .

We will remove 'complete' and 'C', when we use $\overline{\text{span}}\{U\Lambda^*(K) : U \in \mathcal{U}\}$ instead of H . We denote by $g\text{-}\mathcal{NTF}_{\mathcal{U}}(H; K)$ the set of all normalized tight (Parseval) g -frame generators.

Suppose \mathcal{U} is a unitary system on H and $\Lambda \in B(H, K)$ is a \mathcal{U} - g -Bessel generator with respect to K . The analysis operator for \mathcal{U} - g -Bessel generator Λ is denoted by θ_Λ and is defined by

$$\begin{cases} \theta_\Lambda : H \rightarrow l^2(\mathcal{U}, K) \\ \theta_\Lambda f = \{\Lambda U f\}_{U \in \mathcal{U}} \end{cases}$$

and the synthesis operator for \mathcal{U} - g -Bessel generator Λ is denoted by θ_Λ^* . We have

$$S_\Lambda = \theta_\Lambda^* \theta_\Lambda$$

is g -frame operator for \mathcal{U} with respect to K and

$$S_\Lambda f = \sum_{U \in \mathcal{U}} U \Lambda^* \Lambda U f.$$

Example 2.2. Let \mathcal{U} be a unitary system on a Hilbert space H and $\eta \in \mathcal{F}_\mathcal{U}$. Then the functional $\Lambda : H \rightarrow \mathbb{C}$, defined by $\Lambda(x) = \langle x, \eta \rangle$ is a \mathcal{U} - g -frame vector for \mathcal{U} on H with respect to \mathbb{C} .

Example 2.3. Let W be a closed subspace of H which is a fusion frame generator for a unitary system \mathcal{U} with respect to $\{\nu_U\}_{U \in \mathcal{U}}$, i.e. $\{(UW, \nu_U)\}_{U \in \mathcal{U}}$ is a fusion frame (see [16]). Then π_W is a g -frame generator for \mathcal{U} .

The following proposition is a characterization of g -normalized tight frame generators which is a generalization of Proposition 1.1 for \mathcal{U} - g -frames.

Proposition 2.4. Suppose that $\Lambda \in B(H, K)$ is a complete \mathcal{U} - g -wandering generator for \mathcal{U} on H with respect to K and $\Gamma \in B(H, K)$. Then

- i)* $\Gamma \in g\text{-}\mathcal{NTF}_\mathcal{U}(H; K)$ if and only if there exists a unique partial isometry operator $A \in C_\Lambda(\mathcal{U})$ such that $\Lambda A^* = \Gamma$.
- ii)* $\Gamma \in g\text{-}\mathcal{CNTF}_\mathcal{U}(H; K)$ if and only if there exists a unique co-isometry operator $A \in C_\Lambda(\mathcal{U})$ such that $\Lambda A^* = \Gamma$.

Proof . First note that Λ separates $C_\Lambda(\mathcal{U})$. Indeed if $T_1, T_2 \in C_\Lambda(\mathcal{U})$ and $T_1 \neq T_2$, then $\Lambda T_1^* \neq \Lambda T_2^*$. Indeed in contrary if $\Lambda T_1^* = \Lambda T_2^*$, then for every $U \in \mathcal{U}$ we have $\Lambda T_1^* U = \Lambda T_2^* U$ or $\Lambda U T_1^* = \Lambda U T_2^*$. But $\{U \Lambda^*(K) : U \in \mathcal{U}\}$ is a g -orthonormal basis for H with respect to K , and letting $T = T_1 - T_2$ we get $\Lambda U T^* = 0$, or equivalently $T U \Lambda^*(e_k) = 0$, for an orthonormal basis $\{e_k\}_{k \in \mathbb{J}}$ of K . Thus $T = 0$ or $T_1^* = T_2^*$, which is a contradiction. This implies the uniqueness of A . Since otherwise for two different operator $A_1, A_2 \in C_\Lambda(\mathcal{U})$ for which $\Lambda A_1^* = \Gamma$ and $\Lambda A_2^* = \Gamma$, $\Lambda A_1^* \neq \Lambda A_2^*$ that is contradiction. Part *ii)* is guaranteed by *i)*, since

if A is partial isometry and $A \in C_\Lambda(\mathcal{U})$, then $\{U\Gamma^*(K) : U \in \mathcal{U}\} = \{UA\Lambda^*(K) : U \in \mathcal{U}\}$ generates H , if and only if A^* is isometry, so we only prove (i). Suppose Γ is \mathcal{U} - g -normalized tight frame generator for H with respect to K . Define the linear operator B by

$$Bx = \sum_{U \in \mathcal{U}} U\Lambda^*\Gamma Ux,$$

for every $x \in \overline{\text{span}}\{U\Gamma^*(K) : U \in \mathcal{U}\}$, and for every x that $x \perp \overline{\text{span}}\{U\Gamma^*(K) : U \in \mathcal{U}\}$ define $Bx = 0$.

Since Γ is \mathcal{U} - g -normalized tight frame generator for \mathcal{U} , so B is isometry on $\overline{\text{span}}\{U\Gamma^*(K) : U \in \mathcal{U}\}$. Indeed for every $x \in \overline{\text{span}}\{U\Gamma^*(K) : U \in \mathcal{U}\}$

$$\begin{aligned} \|Bx\|^2 &= \langle Bx, Bx \rangle \\ &= \left\langle \sum_{U \in \mathcal{U}} U\Lambda^*\Gamma Ux, \sum_{V \in \mathcal{U}} V\Lambda^*\Gamma Vx \right\rangle \\ &= \sum_{U \in \mathcal{U}} \sum_{V \in \mathcal{U}} \langle U\Lambda^*\Gamma Ux, V\Lambda^*\Gamma Vx \rangle \\ &= \sum_{U \in \mathcal{U}} \langle \Gamma Ux, \Gamma Ux \rangle \\ &= \langle x, x \rangle = \|x\|^2. \end{aligned}$$

Thus B is partial isometry with closed range BH . Let P be the orthogonal projection onto BH and let $A = B^* = B^*P$. We will show that A is partial isometry. First we show that for every $U \in \mathcal{U}$ we have $BUT^* = PU\Lambda^*$. In fact, for every $V \in \mathcal{U}$ and an orthonormal basis $\{e_k\}_{k \in J}$ for K , we have

$$\begin{aligned} \langle BV\Gamma^*e_k, PU\Lambda^*e_j \rangle &= \langle PBV\Gamma^*e_k, U\Lambda^*e_j \rangle \\ &= \langle BV\Gamma^*e_k, U\Lambda^*e_j \rangle \\ &= \left\langle \sum_{W \in \mathcal{U}} W\Lambda^*\Gamma W\Gamma^*e_k, U\Lambda^*e_j \right\rangle \\ &= \sum_{W \in \mathcal{U}} \langle W\Lambda^*\Gamma W\Gamma^*e_k, U\Lambda^*e_j \rangle \\ &= \langle \Gamma U\Gamma^*e_k, e_j \rangle \\ &= \langle V\Gamma^*e_k, U\Gamma^*e_j \rangle. \end{aligned}$$

Since B is isometry on $\overline{\text{span}}\{U\Gamma^*(K); U \in \mathcal{U}\}$, we get

$$\begin{aligned} \langle BV\Gamma^*e_k, PU\Lambda^*e_j \rangle &= \langle V\Gamma^*e_k, U\Gamma^*e_j \rangle \\ &= \langle BV\Gamma^*e_k, BUT^*e_j \rangle \end{aligned}$$

so $PU\Lambda^*e_k = BU\Gamma^*e_k$, or equivalently $PU\Lambda^* = BU\Gamma^*$. We know that Γ is \mathcal{U} - g -normalized tight frame generator for \mathcal{U} so $\overline{\text{span}}\{U\Gamma^*(K) : U \in \mathcal{U}\} = H$, or equivalently $\overline{\text{span}}\{BU\Gamma^*(K) : U \in \mathcal{U}\} = BH$. Also from

$$\langle B\Gamma^*e_k, PU\Lambda^*e_j \rangle = \langle B\Gamma^*e_k, BU\Gamma^*e_j \rangle$$

we get

$$PU\Lambda^*e_j = BU\Gamma^*e_j, \quad (j \in \mathbb{J})$$

which implies that $PU\Lambda^* = BU\Gamma^*$. Now we show that $A = B^*P \in C_\Lambda(\mathcal{U})$ and $\Lambda A^* = \Gamma$. By the above paragraph for every $U \in \mathcal{U}$, $B^*BU\Gamma^* = B^*PU\Lambda^*$. Note that the range of $U\Gamma^*$ is contained in the initial subspace of the partial isometry B so for every $U \in \mathcal{U}$ we have $B^*BU\Gamma^* = U\Gamma^*$. So $U\Gamma^* = U\Lambda A^*$, or equivalently $\Gamma^* = \Lambda A^*$ or $\Lambda A^* = \Gamma$ and $A \in C_\Lambda(\mathcal{U})$. Since $PH(= BH)$ is the final subspace of B and B^* is isometry on PH and takes the value zero on $(PH)^\perp$, thus $B^*P = B^*$ implies that A is a partial isometry. Conversely let A be a partial isometry in $C_\Lambda(\mathcal{U})$ and let $\Lambda A = \Gamma$. Note that A^* is isometric on AH . Then for any $x \in H$, we have

$$\begin{aligned} \|x\|^2 &= \|A^*x\|^2 \\ &= \sum_{U \in \mathcal{U}} \|\Lambda U A^*x\|^2 \\ &= \sum_{U \in \mathcal{U}} \|\Lambda A^*Ux\|^2 \\ &= \sum_{U \in \mathcal{U}} \|\Gamma Ux\|^2. \end{aligned}$$

Thus Γ is a \mathcal{U} - g -normalized tight frame generator for \mathcal{U} on $\overline{\text{span}}\{U\Gamma^*(K); U \in \mathcal{U}\}$. \square

Proposition 2.5. Let \mathcal{S} be a unital semigroup of unitaries in $B(H)$. If \mathcal{S} has a complete \mathcal{U} - g -normalized tight frame generator, then \mathcal{S} is a group.

Proof . Let U be a unitary operator such that $U^{-1} \in \mathcal{S}$. We want to show that $SU^{-1} = \mathcal{S}$. Let $\Gamma \in B(H, K)$ be a complete \mathcal{S} - g -normalized tight frame generator for \mathcal{S} . Then for any $x \in H$, we have

$$\|U^{-1}x\|^2 = \sum_{W \in \mathcal{S}} \|\Gamma W U^{-1}x\|^2 \quad (2.1)$$

and

$$\|U^{-1}x\|^2 = \|x\|^2 = \sum_{W \in \mathcal{S}} \|\Gamma Wx\|^2 \quad (2.2)$$

If $U \notin \mathcal{S}$, then $I \notin SU^{-1}$, and so from (2.1), (2.2) and $SU^{-1} \subseteq \mathcal{S}$ we have $\|\Gamma x\| = 0$, for any $x \in H$, and this means that $\Gamma = 0$, which is a contradiction. So \mathcal{S} is a group. \square Suppose

that Γ and Υ are complete \mathcal{U} - g -normalized tight frame generators for a unitary system \mathcal{U} . If $\{\Gamma U\}_{U \in \mathcal{U}}$ and $\{\Upsilon U\}_{U \in \mathcal{U}}$ are unitarily equivalent, then there is a unitary operator W satisfying $\Gamma U W = \Upsilon U$ for every $U \in \mathcal{U}$. In particular, $\Gamma W = \Upsilon$. Hence $\Gamma U W = \Gamma W U$ for every $U \in \mathcal{U}$. So when \mathcal{U} is a unitary group, we have that $\Gamma \mathcal{U}$ and $\Upsilon \mathcal{U}$ are unitarily equivalent if and only if there is a unitary operator $W \in \mathcal{U}'$ such that $\Gamma W = \Upsilon$. Thus a \mathcal{U} - g -frame unitary equivalence determines an equivalence relation for complete \mathcal{U} - g -normalized tight frame generators. For a set $\mathcal{S} \subseteq B(H)$, we use $W^*(\mathcal{S})$ to denote the Von-Neumann algebra generated by \mathcal{S} , and use \mathcal{S}' to denote the commutant of \mathcal{S} , that is

$$\mathcal{S}' = \{T \in B(H) : SY - YS = 0 \quad ; S \in \mathcal{S}\}.$$

Corollary 2.6. Let \mathcal{U} be a unitary group such that $g\text{-}\mathcal{W}_{\mathcal{U}}(H; K)$ is non-empty. Then every complete \mathcal{U} - g -normalized tight frame generator with respect to K is \mathcal{U} - g -complete wandering generator for \mathcal{U} with respect to K .

Proof . Let $\Lambda \in g\text{-}\mathcal{W}_{\mathcal{U}}(H; K)$ and Γ is a complete \mathcal{U} - g -normalized tight frame generator, by part (ii) of Proposition 2.4, there exists a unique co-isometry $A \in C_{\Lambda}(\mathcal{U}) = \mathcal{U}'$ such that $\Gamma = \Lambda A^*$. Since \mathcal{U}' is finite Von-Neumann algebra (see [14]), so A is unitary. Thus Γ is \mathcal{U} - g -complete wandering generator for \mathcal{U} . \square

By Corollary 2.6 if a unitary group has a \mathcal{U} - g -normalized tight frame generator that is not wandering generator, then $g\text{-}\mathcal{W}_{\mathcal{U}}(H; K)$ is empty. With a minor modification of the proof for Proposition 1.2, one can easily get the following more general result. We leave the details to the interested reader.

Proposition 2.7. Let \mathcal{U} and Λ be as in Proposition 2.4. Then $\Gamma \in B(H, K)$ is a \mathcal{U} - g -frame generator with frame bounds a and b if and only if there exists an (unique) operator $A \in C_{\Lambda}(\mathcal{U})$ such that $\Gamma = \Lambda A^*$ and $aP \leq AA^* \leq bP$ for some orthogonal projection P . In fact P is the orthogonal projection onto $\overline{\text{span}}\{U\Gamma^*(K) : U \in \mathcal{U}\}$. So we have:

Proposition 2.8. $\Gamma \in B(H, K)$ is a complete \mathcal{U} - g -frame generator with frame bounds a and b if and only if there exists an (unique) operator $A \in C_{\Lambda}(\mathcal{U})$ such that

$$aI \leq AA^* \leq bI \quad , \quad \Gamma = \Lambda A^*.$$

The following is a more general characterization for generators.

Theorem 2.9. Let H and K be separable Hilbert spaces and \mathcal{U} be a unitary system on H and $\Lambda \in g\text{-}\mathcal{W}_{\mathcal{U}}(H; K)$ and $\Gamma \in B(H, K)$, then

- i) $\Gamma \in g\text{-}\mathcal{B}_{\mathcal{U}}(H; K)$ if and only if there exists a unique operator $A \in C_{\Lambda}(\mathcal{U})$ such that $\Lambda A^* = \Gamma$.
- ii) $\Gamma \in g\text{-}\mathcal{F}_{\mathcal{U}}(H; K)$ if and only if there exists a unique surjective $A \in C_{\Lambda}(\mathcal{U})$ such that $\Lambda A^* = \Gamma$.
- iii) $\Gamma \in g\text{-}\mathcal{R}_{\mathcal{U}}(H; K)$ if and only if there exists a unique invertible operator $A \in C_{\Lambda}(\mathcal{U})$ such that $\Lambda A^* = \Gamma$.
- iv) $\Gamma \in g\text{-}\mathcal{W}_{\mathcal{U}}(H; K)$ if and only if there exists a unique unitary operator $A \in C_{\Lambda}(\mathcal{U})$ such that $\Lambda A^* = \Gamma$.

Proof . Define $Bx = \sum_{U \in \mathcal{U}} U^* \Lambda^* \Gamma U x$ for every $x \in H$. Let

$$A = B^*. \quad (2.3)$$

For every $x, y \in H$ we have

$$\begin{aligned} \langle B^* x, y \rangle &= \langle x, B y \rangle \\ &= \langle x, \sum_{U \in \mathcal{U}} U^* \Lambda^* \Gamma U y \rangle \\ &= \sum_{U \in \mathcal{U}} \langle x, U^* \Lambda^* \Gamma U y \rangle \\ &= \sum_{U \in \mathcal{U}} \langle U^* \Gamma^* \Lambda U x, y \rangle \\ &= \langle \sum_{U \in \mathcal{U}} U^* \Gamma^* \Lambda U x, y \rangle \end{aligned}$$

or equivalently,

$$Ax = B^* x = \sum_{U \in \mathcal{U}} U^* \Gamma^* \Lambda U x.$$

Also, since $\{\Lambda U\}_{U \in \mathcal{U}}$ is g -orthonormal system, for every $V \in \mathcal{U}$ and $y \in K$ and $x \in H$, we have

$$\begin{aligned} \langle \Lambda V A^* x, y \rangle &= \langle \Lambda V B x, y \rangle \\ &= \langle B x, V^* \Lambda^* y \rangle \\ &= \langle \sum_{U \in \mathcal{U}} U^* \Lambda^* \Gamma U x, V^* \Lambda^* y \rangle \\ &= \sum_{U \in \mathcal{U}} \langle U^* \Lambda^* \Gamma U x, V^* \Lambda^* y \rangle \\ &= \langle \Gamma V x, y \rangle. \end{aligned}$$

Thus $\Lambda V A^* x = \Gamma V x$, setting $V = I$ we have $\Lambda A^* x = \Gamma x$ or equivalently $\Lambda A^* = \Gamma$. Now we prove that $A \in C_{\Lambda}(\mathcal{U})$. For every $U \in \mathcal{U}$ we have

$$\Lambda U A^* f = \Gamma U f = (\Lambda A^*) U f.$$

Furthermore A is unique. Suppose that there is $A' \in C_\Lambda(\mathcal{U})$ is another operator with for which $\Lambda A'^* = \Gamma$, then for every $U \in \mathcal{U}$ we have

$$\Lambda U A'^* = \Lambda A'^* U = \Gamma U = \Lambda A^* U = \Lambda U A^*.$$

Since Λ is \mathcal{U} - g -wandering generator for H with respect to K , thus $A = A'$.
Now for proving every item of this theorem, we define A as (2.3).

Suppose $\Gamma \in g\text{-}\mathcal{B}_\mathcal{U}(H; K)$, so by the first of the proof there exists a unique operator A such that $\Lambda A^* = \Gamma$ and $A \in C_\Lambda(\mathcal{U})$.

Conversely, suppose there exists $A \in C_\Lambda(\mathcal{U})$ such that $\Lambda A^* = \Gamma$, we show that $\Gamma \in g\text{-}\mathcal{B}_\mathcal{U}(H; K)$. Let $x \in H$, we have

$$\begin{aligned} \sum_{U \in \mathcal{U}} \|\Gamma U x\|^2 &= \sum_{U \in \mathcal{U}} \|\Lambda A^* U x\|^2 \\ &= \sum_{U \in \mathcal{U}} \|\Lambda U A^* x\|^2 \\ &= \|A^* x\|^2 \leq \|A^*\|^2 \|x\|^2, \end{aligned}$$

so $\Gamma \in g\text{-}\mathcal{B}_\mathcal{U}(H; K)$. This proves i).

For proving surjectively of A in part ii), it is enough to show that $A^*(= B)$ is injective. Let $Bx = 0$, $x \in H$. For every $V^* \in \mathcal{U}$ and $y \in K$ we have

$$\begin{aligned} 0 &= \langle Bx, V^* \Lambda^* y \rangle = \left\langle \sum_{U \in \mathcal{U}} U^* \Lambda^* \Gamma U x, V^* \Lambda^* y \right\rangle \\ &= \sum_{U \in \mathcal{U}} \langle U^* \Lambda^* \Gamma U x, V^* \Lambda^* y \rangle \\ &= \langle \Gamma V x, y \rangle \\ &= \langle x, V^* \Gamma^* y \rangle. \end{aligned}$$

Since Γ is \mathcal{U} - g -frame generator so $x \perp \overline{\text{span}}\{V^* \Gamma^*(K) : V^* \in \mathcal{U}\} = H$, thus $x = 0$.

Conversely, suppose $A \in C_\Lambda(\mathcal{U})$ is unique such that $\Lambda A^* = \Gamma$. We prove that Γ is \mathcal{U} - g -frame generator for \mathcal{U} . For any $x \in H$,

$$\begin{aligned} \sum_{U \in \mathcal{U}} \|\Gamma U x\|^2 &= \left\langle \sum_{U \in \mathcal{U}} \Gamma U x, \Gamma U x \right\rangle \\ &= \sum_{U \in \mathcal{U}} \langle \Lambda A^* U x, \Lambda A^* U x \rangle \\ &= \sum_{U \in \mathcal{U}} \langle \Lambda U A^* x, \Lambda U A^* x \rangle \\ &= \langle A^* x, A^* x \rangle \leq \|A^*\|^2 \|x\|^2. \end{aligned} \tag{2.4}$$

Also by surjectivity of A we know that B is bounded below, so there exists $C > 0$ such that for any $x \in H$,

$$\begin{aligned}
C\|x\|^2 &\leq \|Bx\|^2 = \langle Bx, Bx \rangle \\
&= \left\langle \sum_{U \in \mathcal{U}} U^* \Lambda^* \Gamma U x, \sum_{V \in \mathcal{U}} V^* \Lambda^* \Gamma V x \right\rangle \\
&= \sum_{U \in \mathcal{U}} \sum_{V \in \mathcal{U}} \langle U^* \Lambda^* \Gamma U x, V^* \Lambda^* \Gamma V x \rangle \\
&= \sum_{U \in \mathcal{U}} \langle \Gamma U x, \Gamma U x \rangle \\
&= \sum_{U \in \mathcal{U}} \|\Gamma U x\|^2. \tag{2.5}
\end{aligned}$$

Now applying (2.4) and (2.5) we conclude that Γ is \mathcal{U} - g -frame generator for \mathcal{U} dependent to K .

For proving iii), it is enough to prove that A is invertible. In ii) we proved that A is surjective. Now we prove that A is also injective. Suppose that $Ax = \sum_{U \in \mathcal{U}} U^* \Gamma^* \Lambda U x = 0$. Since $\Gamma \in g\mathcal{R}_{\mathcal{U}}(H; K)$ so

$$\Lambda U x = 0, \quad (U \in \mathcal{U}).$$

Now since Λ is \mathcal{U} - g -wandering generator so $x = 0$ and so A is injective.

Conversely, suppose that there exists a unique operator $A \in C_{\Lambda}(\mathcal{U})$ such that $\Lambda A^* = \Gamma$. We prove $\Gamma \in g\mathcal{R}_{\mathcal{U}}(H; K)$. We have

$$\Gamma U = \Lambda A^* U = \Lambda U A^*,$$

so by definition of Riesz basis, ΓU is Riesz basis, or $\Gamma \in g\mathcal{R}_{\mathcal{U}}(H; K)$.

Now we prove The last part. We show that A is unitary. By definition of A and the fact that Γ and Λ are \mathcal{U} - g -wandering generators, we get

$$\begin{aligned}
\langle Ax, Ay \rangle &= \left\langle \sum_{U \in \mathcal{U}} U^* \Gamma^* \Lambda U x, \sum_{V \in \mathcal{U}} V^* \Gamma^* \Lambda V y \right\rangle \\
&= \sum_{U \in \mathcal{U}} \sum_{V \in \mathcal{U}} \langle U^* \Gamma^* \Lambda U x, V^* \Gamma^* \Lambda V y \rangle \\
&= \sum_{U \in \mathcal{U}} \langle \Lambda U x, \Lambda U y \rangle = \langle x, y \rangle.
\end{aligned}$$

Thus

$$\langle A^* Ax, y \rangle = \langle x, y \rangle$$

and this means that $A^* A = I$. Similarly we can show that $AA^* = I$, so A is unitary.

For the reverse part, let there exists a (unique) unitary operator $A \in C_{\Lambda}(\mathcal{U})$ such that $\Lambda A^* = \Gamma$. We prove that $\Gamma \in g\mathcal{W}_{\mathcal{U}}(H; K)$. Noting that

$$\Gamma U = \Lambda A^* U = \Lambda U A^*, \quad (U \in \mathcal{U})$$

and by definition of A which is unitary, for every $x, y \in H$, we get

$$\begin{aligned} \langle x, y \rangle &= \langle AA^*x, y \rangle = \langle A^*x, A^*y \rangle \\ &= \left\langle \sum_{U \in \mathcal{U}} U^* \Lambda^* \Gamma U x, \sum_{V \in \mathcal{U}} V^* \Lambda^* \Gamma V y \right\rangle \\ &= \sum_{U \in \mathcal{U}} \sum_{V \in \mathcal{U}} \langle U^* \Lambda^* \Gamma U x, V^* \Lambda^* \Gamma V y \rangle \\ &= \sum_{U \in \mathcal{U}} \langle \Gamma U x, \Gamma U y \rangle. \end{aligned}$$

This implies that $\Gamma \in g\mathcal{W}_{\mathcal{U}}(H; K)$. \square

In the following we show that every \mathcal{U} - g -wandering generator is co-isometry.

Theorem 2.10. Suppose that H and K are two separable Hilbert spaces and \mathcal{U} is unitary system on H and $\Lambda \in g\mathcal{W}_{\mathcal{U}}(H; K)$, then $\Lambda\Lambda^* = I$.

Proof .

By definition of \mathcal{U} - g -wandering generator, for every $U, V \in \mathcal{U}$ and $f, g \in K$, we have

$$\langle (\Lambda U)^* f, (\Lambda U)^* g \rangle = \delta_{U, V} \langle f, g \rangle$$

so

$$\langle U^* \Lambda^* f, U^* \Lambda^* g \rangle = \langle f, g \rangle$$

or equivalently

$$\langle \Lambda U^* U \Lambda^* f, g \rangle = \langle f, g \rangle,$$

which implies that

$$\langle \Lambda \Lambda^* f, g \rangle = \langle f, g \rangle.$$

Thus $\Lambda \Lambda^* = I$. \square

In our last result, we show that the set $g\mathcal{W}_{\mathcal{U}}(H; K)$ is closed under some combinations.

Proposition 2.11. Suppose that H and K are two Hilbert spaces and \mathcal{U} is unitary system on H . Let $\Lambda, \Gamma \in g\mathcal{W}_{\mathcal{U}}(H; K)$ and $V \in C_{\Lambda}(\mathcal{U})$ be a unique unitary operator such that $\Lambda V^* = \Gamma$ and $V^2 = I$, then for every $0 \leq \alpha \leq 2\pi$

$$\cos \alpha \Lambda + i \sin \alpha \Gamma$$

is contained in $g\mathcal{W}_{\mathcal{U}}(H; K)$.

Proof . Suppose $P = \frac{1}{2}(V + I)$. Then P is orthogonal projection in $C_\Lambda(\mathcal{U})$. Define,

$$w_1 = \cos \alpha + i \sin \alpha \quad , \quad w_2 = \cos \alpha - i \sin \alpha.$$

Then $|w_i| = 1$ so $W := w_1P + w_2(I - P)$ is unitary operator that placed in $C_\Lambda(\mathcal{U})$. So $\Lambda W \in g\text{-}\mathcal{W}_\Lambda(H; K)$. But

$$\Lambda W = w_1\Lambda P + w_2\Lambda(I - P)$$

and also

$$\Lambda P = \frac{1}{2}\Lambda V^* + \frac{1}{2}\Lambda = \frac{1}{2}(\Gamma + \Lambda).$$

Similarly $\Lambda(I - P) = \frac{1}{2}(\Lambda - \Gamma)$. Thus

$$\begin{aligned} \Lambda W &= \frac{1}{2}(w_1 + w_2)\Lambda + \frac{1}{2}(w_1 - w_2)\Gamma \\ &= \cos \alpha \Lambda + i \sin \alpha \Gamma. \end{aligned}$$

□

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