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A study on admissible vectors in the quasi-regular representations of generalized Weyl-Heisenberg groups

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Abstract

This paper explores the quasi-regular representation of the generalized Weyl-Heisenberg group, deriving a specific form for an admissible vector within this framework. Moreover, we investigate the square-integrable representations associated with this group, establishing criteria for admissibility and integrability. Finally, illustrative examples are provided to substantiate the theoretical results and clarify the technical aspects of our approach.

Keywords: Generalized Weyl-Heisenberg group, Square integrable, Admissible vector.
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1 Introduction

The study of group representations, especially in the context of Heisenberg and Weyl-Heisenberg groups, has significant implications in mathematics and theoretical physics. The Weyl-Heisenberg group, which naturally arises in quantum mechanics, harmonic analysis, and signal processing, provides a foundational structure for the analysis of time-frequency representations and wavelet transforms. A core aspect of representation theory lies in understanding the conditions under which representations are square-integrable and can be realized through admissible vectors. The admissible vectors facilitate the construction of coherent states, enabling continuous frames in function spaces that have applications in both pure and applied settings (see [2, 3]). However, determining explicit forms for these admissible vectors, particularly in the quasi-regular representations of generalized groups, remains challenging and requires a deep analysis of group structure and integrability conditions.
In this paper, we examine the quasi-regular representation of the generalized Weyl-Heisenberg group and derive explicit conditions and forms for admissible vectors. By analyzing square-integrable representations, we expand on the existing framework of the Weyl-Heisenberg group, contributing new insights into its generalizations. Our results include detailed examples that illustrate the theoretical aspects and validate the practical relevance of our findings.

2 Preliminaries and notation

Let H and K be two locally compact groups with the identity elements e_H and e_K, respectively and let $\tau : H \to Aut(K)$ be a homomorphism such that the map $(h, k) \mapsto \tau_h(k)$ is continuous from $H \times K$ onto K, where $H \times K$ equips with the product topology. The semi- direct product topological group $G_\tau = H \times_\tau K$ is the locally compact topological space $H \times K$ under the product topology, with the group operations:

$$(h_1, k_1) \times_\tau (h_2, k_2) = (h_1h_2, k_1\tau_{h_1}(k_2),$$

$$(h, k)^{-1} = (h^{-1}, \tau_{h^{-1}}(k^{-1})).$$

It is worth to note that $K_1 = \{(e_H, k); k \in K\}$ is a closed normal subgroup and $H_1 = \{(h, e_K); h \in H\}$ is a closed subgroup of G_τ such that $G_\tau = HK$. Moreover, the left Haar measure of the locally compact group G_τ is

$$d\mu_{G_\tau}(h, k) = \delta_H(h)d\mu_H(h)d\mu_K(k),$$

in which $d\mu_H, d\mu_K$ are the left Haar measures on H and K, respectively and $\delta_H : H \to (0, \infty)$ is a positive continuous homomorphism that satisfies

$$d\mu_K(k) = \delta_H(h)d\mu(\tau_h(k)),$$

for $h \in H, k \in K$. Moreover, the modular function Δ_{G_τ} is

$$\Delta_{G_\tau} = \delta_H(h)\Delta_H(h)\Delta_K(k),$$

where Δ_H, Δ_K are the modular functions of H, K, respectively.
When K is also abelian, one can define $\hat{\tau} : H \to Aut(\hat{K})$ via $h \mapsto \hat{\tau_h}$ where

$$\hat{\tau_h}(\omega) = \omega \circ \tau_{h^{-1}},$$

for all $\omega \in \hat{K}$. We usually denote $\omega \circ \tau_{h^{-1}}$ by ω_h. With this notation, it is easy to see

$$\omega_{h_1 h_2} = (\omega_{h_2})_{h_1},$$

where $h_1, h_2 \in H$ and $\omega \in \hat{K}$. The semi-direct product $G_{\hat{\tau}} = H \times_{\hat{\tau}} \hat{K}$ is a locally compact group with the left Haar measure

$$d\mu_{\hat{G}}(h, \omega) = \delta_H(h)^{-1} d\mu_H(h) d\mu_{\hat{K}}(\omega),$$

where $d\mu_{\hat{K}}$ is the Haar measure on $\hat{K}$. Also, for all $h \in H$,

$$d\mu_{\hat{K}}(\omega_h) = \delta_H(h) d\mu_{\hat{K}}(\omega),$$

for $\omega \in \hat{K}$, (see more details in [5, 4, 1, 3]).
Let $G_\tau = H \times_\tau K$, and define $\theta : G_\tau \to Aut(\hat{K} \times \mathbb{T})$ via

$$(h, k) \mapsto \theta_{(h,k)}(\omega, z) = (\hat{\tau_h}(\omega), \hat{\tau_h}(\omega)(k)z) = (\omega_h, \omega_h(k)z),$$

for all $(h, k) \in H \times_\tau K$ and $(\omega, z) \in \hat{K} \times \mathbb{T}$. The mapping θ is a continuous homomorphism. Thus the semi-direct product

$$G_\tau \times_\theta (\hat{K} \times \mathbb{T}) = (H \times_\tau K) \times_\theta (\hat{K} \times \mathbb{T}),$$

is a locally compact group and it is called the generalized Weyl Heisenberg group associated with the semi direct product group $G_\tau = H \times_\tau K$, and denoted by $\mathbb{H}(G_\tau)$. It is easy to see that the group operations of $\mathbb{H}(G_\tau)$ are

$$(h_1, k_1, \omega_1, z_1).(h_2, k_2, \omega_2, z_2) = (h_1 h_2, k_1 \tau_{h_1}(k_2), \omega_1 \omega_{2h_1}, \omega_{2h_1}(k) z_1 z_2),$$

$$(h_1, k_1, \omega_1, z_1)^{-1} = (h_1^{-1}, \tau_{h_1}^{-1}(k^{-1}), \bar{\omega}_{h_1^{-1}}, \bar{\omega}_{h_1^{-1}}(\tau_{h_1}^{-1}(k^{-1}))z^{-1}),$$

for $(h_1, k_1, \omega_1, z_1), (h_2, k_2, \omega_2, z_2) \in \mathbb{H}(G_\tau)$ (see [4]) and the left Haar measure of $\mathbb{H}(G_\tau)$ is:

$$d\mu_{\mathbb{H}(G_\tau)}(h, k, \omega, z) = d\mu_H(h) d\mu_K(k) d\mu_{\hat{K}}(\omega) d\mu_{\mathbb{T}}(z).$$

3 Main results

Throughout this section, we assume that $G_\tau = H \times_\tau K$ is the semi-direct product of H and K that K is abelian. Let $\mathbb{H}(G_\tau) = G_\tau \times_\theta (\hat{K} \times \mathbb{T})$ be the the generalized Weyl Heisenberg group associated with the semi direct product group $G_\tau = H \times_\tau K$.

Now, we are going to define a square integrable representation on $\mathbb{H}(G_\tau)$. With the above notations define $\pi : \mathbb{H}(G_\tau) \to U(L^2(\hat{K}))$ by

$$\pi(h,k,\omega,z)f(\xi) = \delta_H^{-1/2}(h)z\xi(k)\overline{\omega(k)}f((\xi\overline{\omega})_{h^{-1}}), \tag{3.1}$$

then π is a homomorphism. Indeed,

$$\begin{aligned}
&\pi\left((h_1,k_1,\omega_1,z_1)(h_2,k_2,\omega_2,z_2)\right)f(\xi) = \pi(h_1h_2, k_1\tau_{h_1}(k_2), \omega_1(\omega_2)_{h_1}, (\omega_2)_{h_1}(k_1)z_1z_2)f(\xi)\\
&= \delta_H^{-1/2}(h_1h_2)(\omega_2)_{h_1}(k_1)z_1z_2\xi(k_1\tau_{h_1}(k_2))\overline{\omega_1(\omega_2)_{h_1}}(k_1\tau_{h_1}(k_2))f((\xi\overline{\omega_1(\omega_2)_{h_1}})_{(h_1h_2)^{-1}} \quad .\\
&= \delta_H^{-1/2}(h_1h_2)(\omega_2)_{h_1}(k_1)z_1z_2\xi(k_1)\xi_{h_1^{-1}}(k_2)\overline{\omega_1(k_1)(\omega_1)_{h_1^{-1}}(k_2)\omega_2(k_2)}f(\xi_{h_2^{-1}h_1^{-1}}\overline{(\omega_1)}_{h_2^{-1}h_1^{-1}}\overline{(\omega_2)}_{h_2^{-1}})
\end{aligned}$$

Also,

$$\begin{aligned}
&\pi(h_1,k_1,\omega_1,z_1)\pi(h_2,k_2,\omega_2,z_2)f(\xi)\\
&= \delta_H^{-1/2}(h_1)z_1\xi(k_1)\overline{\omega_1}(k_1)\pi(h_2,k_2,\omega_2,z_2)f((\xi\overline{\omega_1})_{h_1^{-1}}\\
&= \delta_H^{-1/2}(h_1)\delta_H^{-1/2}(h_2)z_1z_2\xi(k_1)\overline{\omega_1}(k_1)\overline{\omega_2}(k_2)(\xi\overline{\omega_1})_{h_1^{-1}}(k_2)f((\xi\overline{\omega_1})_{h_1^{-1}}(\overline{\omega_2})_{h_2^{-1}}) \quad .\\
&= \delta_H^{-1/2}(h_1h_2)z_1z_2\xi(k_1)\xi_{h_1^{-1}}(k_2)\overline{\omega_1(k_1)(\omega_1)_{h_1^{-1}}(k_2)\omega_2(k_2)}f(\xi_{h_2^{-1}h_1^{-1}}\overline{(\omega_1)}_{h_2^{-1}h_1^{-1}}\overline{(\omega_2)}_{h_2^{-1}}).
\end{aligned}$$

Moreover, π is unitary. In fact we have,

$$\begin{aligned}
\|\pi(h,k,\omega,z)f\|_2^2 &= \int_{\hat{K}} |\pi(h,k,\omega,z)f(\xi)|^2 d\mu_{\hat{K}}(\xi)\\
&= \int_{\hat{K}} \delta_H^{-1}(h)|f((\xi\overline{\omega})_{h^{-1}}|^2 d\mu_{\hat{K}}(\xi)\\
&= \int_{\hat{K}} \delta_H^{-1}(h)|f((\xi)_{h^{-1}}|^2 d\mu_{\hat{K}}(\xi) \quad .\\
&= \int_{\hat{K}} f((\xi)|^2 d\mu_{\hat{K}}(\xi)\\
&= \|f\|_2^2.
\end{aligned}$$

And it is easy to check that π is continuous and onto. So, π is a continuous unitary representation of group $\mathbb{H}(G_\tau)$ to the Hilbert space $L^2(\hat{K})$. In the sequel, we show that π is irreducible when H is compact. Furthermore, it is also shown that π is square integrable if and only if H is compact. Note that when H is a compact group, we normalize the Haar measure μ_H such that $\mu_H(H) = 1$.

Theorem 3.1. Let $\mathbb{H}(G_\tau) = (H \times_\tau K) \times_\theta (\hat{K} \times \mathbb{T})$ where H is a locally compact group and K is a locally compact abelian group. Then for φ, ψ in $L^2(\hat{K})$,

$$\int_{\mathbb{H}(G_\tau)} |\prec \varphi, \pi(h,k,\omega,z)\psi \succ|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z) = \|\varphi\|_2^2\|\psi\|_2^2. \tag{3.2}$$

if and only if H is compact.

Proof . For φ, ψ in $L^2(\hat{K})$ we first consider the following observations:

$$\begin{aligned}
&\int_{\mathbb{H}(G_\tau)} | \prec \varphi, \pi(h,k,\omega,z)\psi \succ |^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)\\
&= \int_{\mathbb{H}(G_\tau)} | \int_{\hat{K}} \varphi(\xi)\overline{\pi(h,k,\omega,z)\psi(\xi)} d\mu_{\hat{K}}(\xi)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)\\
&= \int_{\mathbb{H}(G_\tau)} | \int_{\hat{K}} \varphi(\xi)\delta_H^{-1/2}(h)\overline{z\xi(k)}\omega(k)\overline{\psi(\xi\overline{\omega})_{h^{-1}}} d\mu_{\hat{K}}(\xi)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)\\
&= \int_{\mathbb{H}(G_\tau)} | \int_{\hat{K}} \varphi(\xi\omega)\delta_H^{-1/2}(h)\overline{z\xi(k)\psi(\xi)_{h^{-1}}} d\mu_{\hat{K}}(\xi)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)\\
&= \int_{\mathbb{H}(G_\tau)} | \int_{\hat{K}} R_\omega\varphi(\xi)\delta_H^{-1/2}(h)\overline{z\xi(k)\psi(\xi\circ\tau_h)} d\mu_{\hat{K}}(\xi)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)\\
&= \int_{\mathbb{H}(G_\tau)} | \int_{\hat{K}} R_\omega\varphi(\xi\circ\tau_{h^{-1}})\delta_H^{-1/2}(h)\overline{z\xi\circ\tau_{h^{-1}}(k)\psi(\xi)} d\mu_{\hat{K}}(\xi_h)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)\\
&= \int_{\mathbb{H}(G_\tau)} | \int_{\hat{K}} R_\omega\varphi(\xi\circ\tau_{h^{-1}})\delta_H^{1/2}(h)\overline{z\xi(\tau_{h^{-1}}(k))\psi(\xi)} d\mu_{\hat{K}}(\xi)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)\\
&= \int_{\mathbb{H}(G_\tau)} \delta_H(h) | \int_{\hat{K}} (R_\omega\varphi(.\circ\tau_{h^{-1}}).\overline{\psi})(\xi)\overline{\xi(\tau_{h^{-1}}(k))} d\mu_{\hat{K}}(\xi)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)\\
&= \int_{\mathbb{H}(G_\tau)} \delta_H(h) |(\widehat{R_\omega\varphi(.\circ\tau_{h^{-1}}).\overline{\psi}})(\tau_{h^{-1}}(k))|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z)\\
&= \int_H \delta_H(h) \int_{\hat{K}} \int_K |(\widehat{R_\omega\varphi(.\circ\tau_{h^{-1}}).\overline{\psi}})(\tau_{h^{-1}}(k))|^2 d\mu_K(k) d\mu_{\hat{K}}(\omega) d\mu_H(h)\\
&= \int_H \int_{\hat{K}} \int_K |(\widehat{R_\omega\varphi(.\circ\tau_{h^{-1}}).\overline{\psi}})(k)|^2 d\mu_K(k) d\mu_{\hat{K}}(\omega) d\mu_H(h)\\
&= \int_H \int_{\hat{K}} \int_{\hat{K}} |(R_\omega\varphi(.\circ\tau_{h^{-1}}).\overline{\psi})(\xi)|^2 d\mu_{\hat{K}}(\xi) d\mu_{\hat{K}}(\omega) d\mu_H(h)\\
&= \int_H \int_{\hat{K}} \int_{\hat{K}} |R_\omega\varphi(\xi\circ\tau_{h^{-1}}).\overline{\psi}(\xi)|^2 d\mu_{\hat{K}}(\xi) d\mu_{\hat{K}}(\omega) d\mu_H(h)\\
&= \int_H \int_{\hat{K}} \int_{\hat{K}} \delta_H(h) |R_\omega\varphi(\xi).\overline{\psi}(\xi\circ\tau_h)|^2 d\mu_{\hat{K}}(\xi) d\mu_{\hat{K}}(\omega) d\mu_H(h)\\
&= \int_H \int_{\hat{K}} \|\varphi\|_2^2 \delta_H(h) |\overline{\psi}(\xi\circ\tau_h)|^2 d\mu_{\hat{K}}(\xi) d\mu_H(h)\\
&= \|\varphi\|_2^2 \|\psi\|_2^2 \mu_H(H).
\end{aligned}$$

Now, if H is compact, then $\mu_H(H) = 1$. So, (3.2) holds. Conversely, if (3.2) holds, the above observation implies that $\mu_H(H) = 1$. So, we can conclude that H is compact. □

Corollary 3.2. With notation as above, the representation π of $\mathbb{H}(G_\tau)$ on $L^2(\hat{K})$ is irreducible if H is compact.

Proof . If H is compact, then (3.2) in Theorem 3.1 holds. Now, suppose that M is a closed subspace of the Hilbert space $L^2(\hat{K})$ that is invariant under π. Then for any $\varphi \in M$ we have,

$$\{\pi(h,k,\omega,z)\varphi; \quad (h,k,\omega,z) \in \mathbb{H}(G_\tau)\} \subseteq M.$$

Let $\psi \in L^2(\hat{K})$ be orthogonal to M, that is $\prec \psi, \pi(h,k,\omega,z)\varphi \succ = 0$, for all $(h,k,\omega,z) \in \mathbb{H}(G_\tau)$. Thus by (3.2), $\|\varphi\|_2\|\psi\|_2 = 0$, and hence $\psi = 0$. So, $M^\perp = \{0\}$, that is, $M = L^2(\hat{K})$. Namely, π is irreducible. □

We remind the reader that, an irreducible representation π of $\mathbb{H}(G_\tau)$ on $L^2(\hat{K})$ is called square integrable if there exists a non zero element ψ in $L^2(\hat{K})$ such that

$$\prec \pi(.,.,.,.)\psi, f \succ \in L^2(\mathbb{H}(G_\tau)), \tag{3.3}$$

for all $f \in L^2(\hat{K})$. A unit vector ψ satisfying (3.3) is said to be an admissible wavelet for π, and the constant

$$c_\psi = \int_{\mathbb{H}(G_\tau)} | \prec \pi(h,k,\omega,z)\psi, \psi \succ |^2 d\mu_{\mathbb{H}(G_\tau)},$$

is called the wavelet constant associated to the admissible wavelet ψ.
Also, for the admissible wavelet ψ, the continuous wavelet transform is defined by

$$W_\psi f(h,k,\omega,z) = \prec f, \pi(h,k,\omega,z)\psi \succ .$$

It is easy to see that $(h,k,\omega,z) \mapsto W_\psi f(h,k,\omega,z)$ is a continuous function on $\mathbb{H}(G_\tau)$. Moreover, W_ψ intertwines π and the left regular representation on $\mathbb{H}(G_\tau)$.

Corollary 3.3. The representation π of the GWH group $\mathbb{H}(G_\tau) = (H \times_\tau K) \times_\theta (\hat{K} \times \mathbb{T})$ on $L^2(\hat{K})$ is square integrable if and only if H is compact.

Proof . If H is compact, then by Theorem 3.1 and Corollary 3.2, π is square integrable. For the inverse, if π is square integrable, then there exists a non zero element $\varphi \in L^2(\hat{K})$ such that

$$\prec \pi(.,.,.,.)\varphi, \psi \succ \in L^2(\mathbb{H}(G_\tau)),$$

for all $\psi \in L^2(\hat{K})$. On the other hand,

$$\int_{\mathbb{H}(G_\tau)} | \prec \varphi, \pi(h,k,\omega,z)\psi \succ |^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z) = \|\varphi\|_2^2 \|\psi\|_2^2 \mu_H(H).$$

So $\mu_H(H) < \infty$. That is H is compact. $\square$

Remark 3.4. There is another irreducible representation of $\mathbb{H}(G_\tau)$ on Hilbert space $L^2(K)$. Indeed, consider

$$\tilde{\pi} : \mathbb{H}(G_\tau) \to U(L^2(K)), \qquad \tilde{\pi}(h,k,\omega,z)f(k') = \delta_H(h)^{1/2} z\omega(k') f(\tau_{h^{-1}}(k'k)),$$

for all $(h,k,\omega,z) \in \mathbb{H}(G_\tau), f \in L^2(K)$. $\tilde{\pi}$ is homomorphism and unitary. In fact we have

$$\begin{aligned}
&\tilde{\pi}((h_1,k_1,\omega_1,z_1)(h_2,k_2,\omega_2,z_2))f(k') \\
&= \tilde{\pi}(h_1h_2, k_1\tau_{h_1}(k_2), \omega_1(\omega_2)_{h_1}, (\omega_2)_{h_1}(k_1)z_1z_2)f(k') \\
&= \delta_H^{1/2}(h_1h_2)(\omega_2)_{h_1}(k_1)z_1z_2\omega_1(\omega_2)_{h_1}(k')f(\tau_{(h_1h_2)^{-1}}(k'(k_1\tau_{h1}(k_2)) \\
&= \delta_H^{1/2}(h_1h_2)z_1z_2\omega_1(k')(\omega_2)_{h_1}(k'k_1)f(\tau_{h_2^{-1}h_1^{-1}}(k'k_1\tau_{h_1}(k_2)) \\
&= \delta_H^{1/2}(h_1h_2)z_1z_2\omega_1(k')\omega_2)_{h_1}(k'k_1)f(\tau_{h_2^{-1}h_1^{-1}}(k'k_1)\tau_{h_2^{-1}}(k_2)),
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\pi}(h_1,k_1,\omega_1,z_1)\tilde{\pi}(h_2,k_2,\omega_2,z_2)f(k')\\
&=\delta_H^{1/2}(h_1)z_1\omega_1(k')\tilde{\pi}(h_2,k_2,\omega_2,z_2)f(\tau_{h_1^{-1}}(k'k_1))\\
&=\delta_H^{1/2}(h_1)\delta_H^{1/2}(h_2)z_1z_2\omega_1(k')\omega_2(\tau_{h_1^{-1}}(k'k_1))f(\tau_{h_2^{-1}}(\tau_{h_1^{-1}}(k'k_1)k_2)\\
&=\delta_H^{1/2}(h_1h_2)z_1z_2\omega_1(k')(\omega_2)_{h_1}(k'k_1)f(\tau_{h_2^{-1}h_1^{-1}}(k'k_1)\tau_{h_2^{-1}}(k_2)).
\end{aligned}$$

Also,

$$\begin{aligned}
\|\tilde{\pi}(h,k,\omega,z)f\|_2^2 &= \int_K |\tilde{\pi}(h,k,\omega,z)f(k')|^2 d\mu_K(k')\\
&= \int_K \delta_H(h)|f(\tau_{h^{-1}}(k'k))|^2 d\mu_K(k')\\
&= \int_K \delta_H(h)|f(k')|^2 d\mu_K(\tau_h(k'))\\
&= \int_K |f(k')|^2 d\mu_K(k')\\
&= \|f\|_2^2.
\end{aligned}$$

Using the Plancherel theorem, $\pi, \tilde{\pi}$ are unitarily equivalent. So, $\tilde{\pi}$ is square integrable if and only if π is square integrable.

Remark 3.5. The inverse of Corollary 3.2 does not hold, in general. An obvious example is, in case H is a non compact group and K is the trivial group $\{e\}$. Then the representation $\pi : \mathbb{H}(H \times_\tau \{e\}) \to U(\mathbb{C})$ is an irreducible representation. Here we give a non trivial example in which π is an irreducible representation, but H is not compact. Let $H = \mathbb{R}^+, K = \mathbb{R}$. Define the representation π of $\mathbb{H}(\mathbb{R}^+ \times_\tau \mathbb{R})$ as follows:

$$\pi : \mathbb{H}(\mathbb{R}^+ \times_\tau \mathbb{R}) \to U(L^2(\mathbb{R})); \qquad \pi(a,x,\omega,z)f(\xi) = a^{1/2} z e^{2\pi i x(\xi-\omega)} f((\bar{\xi\omega})_{a^{-1}}),$$

in which $(\bar{\xi\omega})_{a^{-1}} = (\bar{\xi\omega}) \circ \tau_a$, $\tau_a(x) = a.x$ and $\delta_H(a) = a^{-1}$. This representation is irreducible. Indeed, let M be a closed invariant subspace of $L^2(\mathbb{R})$ under π. Then for any $f \in M$, we have $\pi(h,k,\omega,z)f \in M$. Consider $0 \neq g \in M^\perp$, so that $\prec g, \pi(h,k,\omega,z)f \succ = 0$. Then

$$0 = \int_{\mathbb{R}} g(\xi) e^{-2\pi i x \xi} \bar{f}((\xi\bar{\omega})_{a^{-1}}) d\xi = \int_{\mathbb{R}} g(\xi_a \omega) e^{-2\pi i x \xi_a \omega} \bar{f}(\xi) d\xi.$$

Thus, $g(\xi_a\omega)\bar{f}(\xi) = 0$, for almost all $\xi \in \mathbb{R}$. Suppose that $\bar{f}(\xi) \neq 0$, for all ξ in a set A with positive measure. Then for all $\xi \in A$, $g(\xi_a\omega) = 0$, for all $\omega \in \mathbb{R}, a \in \mathbb{R}^+$. Thus $g = 0$. This is a contradiction. So, π is an irreducible representation, but H is not compact.

In the sequel, we define the quasi regular representation and we obtain a concrete form for an admissible vector. Note that $\mathbb{H}(G_\tau)$ acts on the Hilbert space $L^2(\hat{K} \times \mathbb{T})$ and this action induces the quasi regular representation $\{\rho, L^2(\hat{K} \times \mathbb{T})\}$ as follows:

$$\rho : (H \times_\tau K) \times_\theta (\hat{K} \times \mathbb{T}) \to U(L^2(\hat{K} \times \mathbb{T})), \tag{3.4}$$

where

$$\begin{aligned}\rho(h,k,\omega,z)f(\xi,t) &= \delta^{1/2}_{H\times_\tau K}(h,k)f(\theta_{(h,k)^{-1}}(\xi,t)(\omega,z)^{-1})\\ &= \delta_H^{-1/2}(h)f(\theta_{(h^{-1},\tau_{h^{-1}}(k^{-1})}(\xi\bar{\omega},tz^{-1}))\\ &= \delta_H^{-1/2}(h)f((\xi\bar{\omega})_{h^{-1}},(\xi\bar{\omega})_{h^{-1}}(\tau_{h^{-1}}(k^{-1})).tz^{-1}).\end{aligned}$$

Note that $\delta_{H\times_\tau K}(h,k)=\delta_H(h)^{-1}$ (see Corollary 3.3 in [4]).

A type of the Fourier transform of the quasi regular representation ρ obtains as follows:
$\rho(h,\widehat{k,\omega,}z)f(k',n')$
$=\int_{\hat{K}\times\mathbb{T}}\rho(h,k,\omega,z)f(\xi,t)\overline{(k',n')(\xi,t)}d\mu_{\hat{K}}(\xi)d\mu_{\mathbb{T}}(t)$
$=\delta_H(h)^{-1/2}\int_{\hat{K}\times\mathbb{T}}f((\xi\overline{\omega})_{h^{-1}},(\xi\overline{\omega})_{h^{-1}}(\tau_{h^{-1}}(k^{-1}))tz^{-1})\overline{\xi(k')t^{n'}}d\mu_{\hat{K}}(\xi)d\mu_{\mathbb{T}}(t)$
$=\delta(h)^{-1/2}\overline{z^{n'}}\int_{\hat{K}\times\mathbb{T}}f(\xi)_{h^{-1}},\xi_{h^{-1}}(\tau_{h^{-1}}(k^{-1}))t)\overline{\omega}(k')\xi(\bar{k}')\overline{t^{n'}}d\mu_{\hat{K}}(\xi)d\mu_{\mathbb{T}}(t)$
$=\delta_H(h)^{-1/2}z^{\bar{n}'}\bar{\omega}(k')\int_{\hat{K}\times\mathbb{T}}f(\theta_{(h^{-1},\tau_{h^{-1}})}(\xi,t)\xi(\bar{k}')t^{\bar{n}'}d\mu_{\hat{K}}(\xi)d\mu_{\mathbb{T}}(t)$
$=\delta_H(h)^{-1/2}z^{\bar{n}'}\bar{\omega}(k')\int_{\hat{K}\times\mathbb{T}}f\circ\theta_{(h,k)^{-1}}(\xi,t)\overline{(k',n')(\xi,t)}d\mu_{\hat{K}}(\xi)d\mu_{\mathbb{T}}(t)$
$=\delta_H(h)^{-1/2}z^{\bar{n}'}\bar{\omega}(k')(f\circ\widehat{\theta_{(h,k)^{-1}}})(k',n'),$
for all $(k',n')\in K\times\mathbb{Z}=(\widehat{\hat{K}\times\mathbb{T}})$. So,

$$\rho(h,\widehat{k,\omega,}z)f(k',n')=\delta_H(h)^{-1/2}z^{\bar{n}'}\bar{\omega}(k')(f\circ\widehat{\theta_{(h,k)^{-1}}})(k',n'). \tag{3.5}$$

Theorem 3.6. With the notation as above, let ρ be the quasi regular representation on $\mathbb{H}(G_\tau)$, and $\psi,f\in L^2(\hat{K}\times\mathbb{T})$.

(i) If ψ is a admissible vector, then

$$W_\psi f(h,k,\omega,z)=\delta_H^{-1/2}(h)\int_K\sum_{n'\in\mathbb{Z}}\hat{f}(k',n')z^{n'}\omega(k')\overline{(\widehat{\psi\circ\theta})}_{(h,k)^{-1}}(k',n')d\mu_K(k').$$

(ii) The vector ψ is admissible if

$$\int_{H\times_\tau K}|\hat{\psi}(k',n')\circ\theta_{(h,k)^{-1}}|^2d\mu_{H\times K}(h,k)<\infty.$$

Proof . For $(k',n')\in K\times\mathbb{Z}$,

(i) By the Plancherel's theorem and (3.5), we have

$$\begin{aligned} W_\psi f(h,k,\omega,z) &= \prec f, \rho(h,k,\omega,z)\psi \succ \\ &= \prec \hat{f}, \rho(h,\widehat{k,\omega},z)\psi \succ \\ &= \delta_H^{-1/2}(h)\int_K \sum_{n'\in\mathbb{Z}} \hat{f}(k',n')z^{n'}\omega(k')\overline{\widehat{(\psi\circ\theta)}}_{(h,k)^{-1}}(k',n')d\mu_K(k'). \end{aligned}$$

(ii) By applying the part (i), for $f \in L^2(\hat{K}\times\mathbb{T})$, we get
$\int_{\hat{K}\times\mathbb{T}} |W_\psi f(h,k,\omega,z)|^2 d\mu_{\hat{K}\times\mathbb{T}}(\omega,z)$
$= \int_{\hat{K}\times\mathbb{T}} W_\psi f(h,k,\omega,z)\overline{W_\psi f(h,k,\omega,z)} d\mu_{\hat{K}\times\mathbb{T}}(\omega,z)$
$= \underline{\delta_H^{-1}(h)\int_{\hat{K}\times\mathbb{T}}[(\int_K \sum_{n'\in\mathbb{Z}} \hat{f}(k',n')z^{n'}\omega(k')\overline{\widehat{(\psi\circ\theta)}}_{(h,k)^{-1}}(k',n')d\mu_K(k'))}$
$\times(\int_K \sum_{n''\in\mathbb{Z}} \hat{f}(k',n')z^{n''}\omega(k'')\overline{\widehat{(\psi\circ\theta)}}_{(h,k)^{-1}}(k'',n'')d\mu_K(k''))]$
$= \delta_H^{-1}(h)\int_{\hat{K}\times\mathbb{T}} |\hat{F}(\omega,z)|^2 d\mu_{\hat{K}\times\mathbb{T}}$
$= \delta_H^{-1}(h)\int_{\hat{K}\times\mathbb{T}} |F(k',n')|^2 d\mu_{K\times\mathbb{Z}}$
$= \delta_H^{-1}(h)\int_K \sum_{n'\in\mathbb{Z}} |\hat{f}(k',n')|^2|\widehat{(\psi\circ\theta)}(k',n')|^2 d\mu_K(k')$,
where $\hat{F} = \hat{f}(\psi \hat{\circ} \theta) \in L^1(K\times\mathbb{Z})$. It is easy to see that

$$\widehat{(\psi\circ\theta)}((k',n') = \delta_H^{-1}(h)\hat{\psi}(k',n')\circ\theta_{(h,k)^{-1}}.$$

Then

$$\int_{\hat{K}\times\mathbb{T}} |W_\psi f(h,k,\omega,z)|^2 d\mu_{\hat{K}\times\mathbb{T}}(\omega,z) = \delta_H^{-1}(h)\int_K \sum_{n'\in\mathbb{Z}} |\hat{f}(k',n')|^2|(\widehat{\psi(k',n'})\circ\theta_{(h,k)^{-1}}|^2 d\mu_K(k'). \tag{3.6}$$

Now, by using (3.6) we have

$$\begin{aligned} \|W_\psi f\|_2^2 &= \int_{\mathbb{H}(G_\tau)} |W_\psi f(h,k,\omega,z)|^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z) \\ &= \int_{H\times_\tau K}\int_{\hat{K}\times\mathbb{T}} |W_\psi f(h,k,\omega,z)|^2 \delta_H^{-1}(h) d\mu_{\hat{K}\times\mathbb{T}}(\omega,z) d\mu_{H\times_\tau K}(h,k) \\ &= \int_{H\times_\tau K}\int_K \sum_{n'\in\mathbb{Z}} |\hat{f}(k',n')|^2|(\widehat{\psi(k',n'})\circ\theta_{(h,k)^{-1}}|^2 d\mu_K(k') d\mu_{H\times_\tau K}(h,k) \\ &= \|f\|_2^2 \int_{H\times_\tau K} |(\widehat{\psi(k',n'})\circ\theta_{(h,k)^{-1}}|^2 d\mu_{H\times_\tau K}(h,k), \end{aligned}$$

and then the proof of part (ii) is complete. □

Here, we give some examples to support our technical considerations.

Example 3.7. Let K be an abelian locally compact group and $H = \{e\}$ (the trivial group). In this case the generalized Weyl-Heisenberg group $\mathbb{H}(G_\tau)$ coincides with the standard Weyl-Heisenberg group $G := K \times_\theta (\hat{K}\times\mathbb{T})$. In this case the square integrable representation of

$G = K \times_\theta (\hat{K} \times \mathbb{T})$ on $L^2(\hat{K})$ is as follows:

$$\pi(k, \omega, z) f(\xi) = z\xi(k)\overline{\omega(k)} f(\xi\overline{\omega}). \tag{3.7}$$

Example 3.8. Let $E(n)$ be the Euclidean group which is the semi-direct product of $So(n) \times_\tau \mathbb{R}^n$ where the continuous homomorphism $\tau : So(n) \to Aut(\mathbb{R}^n)$ given by $\sigma \mapsto \tau_\sigma$ via $\tau_\sigma(x) = \sigma x$, for all $x \in \mathbb{R}^n$. The group operation for $E(n)$ is

$$(\sigma_1, x_1) \times_\tau (\sigma_2, x_2) = (\sigma_1\sigma_2, x_1 + \sigma_1 x_2).$$

Consider the continuous homomorphism $\hat{\tau} : So(n) \to Aut(\mathbb{R}^n)$ via $\sigma \mapsto \hat{\tau_\sigma}$ which is given by $\hat{\tau_\sigma}((\omega) = \omega_\sigma = \omega \circ \tau_{\sigma^{-1}}$. Thus the generalized Weyl-Heisenberg group of $E(n)$, is the set $\mathbb{H}(E(n)) = (So(n) \times_\tau \mathbb{R}^n) \times_\theta (\mathbb{R}^n \times \mathbb{T})$ with the group operation

$$(\sigma_1, x_1, \omega_1, z_1)(\sigma_2, x_2, \omega_2, z_2) = (\sigma_{\sigma_2}, x_1 + \sigma_1 x_2, \omega_1(\omega_2)_{\sigma_1}, (\omega_2)_{\sigma_1}(x_1) z_1 z_2),$$

for all $(\sigma_1, x_1, \omega_1, z_1)(\sigma_2, x_2, \omega_2, z_2) \in \mathbb{H}(E(n))$ and with the product topology. Then the square integrable representation π of $\mathbb{H}(E(n))$ onto $L^2(\mathbb{R}^n)$ is

$$\pi(\sigma, x, \omega, z) f(\xi) = e^{2\pi i x(\xi-\omega)} f((\xi - \omega)_{\sigma^{-1}}).$$

Note that H is compact and $\delta_H(h) = 1$.

Example 3.9. Let $\mathbb{H}(\mathbb{R}^n) = \mathbb{R}^n \times_\theta (\mathbb{R}^n \times \mathbb{T})$ be the classical Heisenberg group on $\mathbb{R}^n$, in which the continuous homomorphism $x \mapsto \theta_x$ from $\mathbb{R}^n$ into $Aut(\mathbb{R}^n \times \mathbb{T})$ is defined by $\theta_x(y, z) = (y, ze^{2\pi i x.y})$. Then the square integrable representation π of $\mathbb{H}(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$ is

$$\pi(x, \omega, z) f(\xi) = z.e^{2\pi i x(\xi-\omega)} f(\xi - \omega).$$

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