Journal of Frame and Matrix Theory (JFMT) 2 (2025) No. 1, 70–80
Online ISSN: 3041-8526
Published by Faculty of Mathematics and Computer Science
Hakim Sabzevari University
http://dx.doi.org/10.22034/jfmt.2025.494407.1022

Generalized multipliers of controlled sequences

Hessam Hosseinnezhad[a]

[a] *Department of Basic Sciences, Shahid Sattari Aeronautical University of Science and Technology, Tehran, Iran*

(Communicated by Mehdi Zaferanieh)

Abstract

In this paper, we introduce a general version of multipliers for controlled sequences. In fact, by combining analysis, an operator on the Hilbert space $\ell^2(\mathbb{I})$ and synthesis, we reach so-called generalized controlled Bessel multipliers. Some basic properties of this class of operators are investigated. Specially, we are interested to determine cases when generalized multipliers are invertible. Subsequently, our attention is on how to express the inverse of an invertible generalized frame multiplier as a multiplier.

Keywords: Controlled frames, Controlled Riesz bases, Bessel multipliers.
MSC 2010: Primary: 42C15; Secondary: 46C05.

1 Introduction

The concept of frames (classical version) for a Hilbert space introduced firstly by Duffin and Schaeffer in [9], which has already been applied in various fields because of its flexibility

Email address: hosseinnezhad@ssau.ac.ir (Hessam Hosseinnezhad)

Received: 16 December 2024 *Accepted:* 17 March 2025

compared with a basis. In the past years, many attention was paid to the extension of frames. One of the last extensions of frames is weighted and controlled frames that introduced by P. Balazs, J. P. Antoine and A. Grybos to improve the numerical efficiency of iterative algorithms for inverting the frame operator [5].

In 1960, Schatten [16] studied operators of the form

$$\sum_{i\in\mathbb{I}} \lambda_i (x_i \otimes \bar{y}_i), \tag{1.1}$$

where $\{x_i\}_{i\in\mathbb{I}}$ and $\{y_i\}_{i\in\mathbb{I}}$ are orthonormal sequences in a Hilbert space. It is then showed that every compact operator is of the form in (1.1), with $\lambda_i \geq 0$, and $\lambda_i \to 0$ as $i \to \infty$ (the spectral theorem for compact operators). In 2007, P. Balazs [3] generalized this by replacing $\{x_i\}_{i\in\mathbb{I}}$ and $\{y_i\}_{i\in\mathbb{I}}$ by Bessel sequences. These are operators that combine analysis, a multiplication with a fixed sequence (called the symbol) and synthesis. Several basic properties of these operators were investigated in [3]. Recently, the concept of multipliers has been extended and introduced for continuous frames [7]], fusion frames [2], p-Bessel sequences [14], generalized frames [13], controlled frames [15], Hilbert C^*-modules [10] and etc. The notion of generalized multipliers appeared firstly in [1], by replacing the fixed multiplication operator by a bounded and linear operator on $\ell^2(\mathbb{I})$. This class of operators is not only of interest for applications in modern life, for example in acoustics [17], psychoacoustics [6] and denoising [12], but also it is important in different branches of functional analysis [4]. In this respect, it is important to find the inverse of a multiplier if it exists.

In the present paper, the concept of generalized multipliers is extended for controlled sequences and then, some properties of these operators are investigated in more details. In particular, special attention is devoted to the study of invertible generalized multipliers.

2 Notation and preliminaries

In this section, we collect the basic notation and some preliminary results. Throughout the paper, $\mathcal{H}$ is a separable Hilbert space and $\mathbb{I}$ is an at most countable index set. Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded and linear operators on $\mathcal{H}$. For $U \in \mathcal{B}(\mathcal{H})$, the notations U^* is the adjoint operator of U. We define $\mathcal{GL}(\mathcal{H})$ as the set of all bounded and linear operators on $\mathcal{H}$ with a bounded inverse. Given $0 < p < \infty$, we define the Schatten p-class of $\mathcal{H}$, denoted $\mathcal{S}_p(\mathcal{H})$, as the space of all compact operators U on $\mathcal{H}$ for which singular value sequence $\{\lambda_i\}_{i\in\mathbb{I}}$ belongs to $\ell^2(\mathbb{I})$. It is proved that $\mathcal{S}_p(\mathcal{H})$ is a two sided $*$-ideal of $\mathcal{B}(\mathcal{H})$.

Definition 2.1. Let $C, C' \in \mathcal{GL}(\mathcal{H})$ and $F = \{f_i\}_{i\in\mathbb{I}}$ be a sequence in $\mathcal{H}$. Then, F is called a (C, C')-controlled frame if there exist two constants $0 < A \leq B < \infty$ such that for every $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} \langle f, Cf_i\rangle \langle C' f_i, f\rangle \leq B\|f\|^2. \tag{2.1}$$

If only the right inequality in (2.1) holds, then we call F a (C, C')-controlled Bessel sequence. We denote the (C, C)-controlled Bessel sequence and (C, C)-controlled frame by C^2-controlled Bessel sequence and C^2-controlled frame, respectively.

Let $F = \{f_i\}_{i\in\mathbb{I}}$ be a C^2-controlled Bessel sequence in $\mathcal{H}$. Then, the analysis operator $T_{CF} : \mathcal{H} \to \ell^2(\mathbb{I})$ is defined as

$$T_{CF}f := \{\langle f, Cf_i\rangle\}_{i\in\mathbb{I}}, \quad (f \in \mathcal{H}).$$

Clearly, the operator T_{CF} is linear and bounded with $\|T_{CF}\| \leq \sqrt{B}$. The adjoint operator of T_{CF}, which is called the synthesis operator, is defined as

$$T^*_{CF} : \ell^2(\mathbb{I}) \to \mathcal{H}, \;\; T^*_{CF}\{a_i\}_{i\in\mathbb{I}} := \sum_{i\in\mathbb{I}} a_i C f_i, \;\; (\{a_i\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})).$$

For a (C, C')-controlled Bessel sequence $F = \{f_i\}_{i\in\mathbb{I}}$, by composing the analysis and synthesis operators $T_{C'F}$ and T^*_{CF}, we get the controlled frame operator $S_{CFC'} : \mathcal{H} \to \mathcal{H}$ as

$$Sf := \sum_{i\in\mathbb{I}} \langle f, C'f_i\rangle Cf_i. \tag{2.2}$$

Clearly, if $F = \{f_i\}_{i\in\mathbb{I}}$ is a (C, C')-controlled frame, then $S_{CFC'}$ is a positive and invertible operator.

Remark 2.2. Assume that $F = \{f_i\}_{i\in\mathbb{I}}$ is a C^2-controlled Bessel sequence in $\mathcal{H}$. Consider the following operator

$$T^*_{F} : \ell^2(\mathbb{I}) \to \mathcal{H}, \;\; T^*_{F}\{a_i\}_{i\in\mathbb{I}} := \sum_{i\in\mathbb{I}} a_i f_i, \;\; (\{a_i\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})).$$

Then, for every $\{a_i\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})$, we have

$$T^*_{CF}\{a_i\}_{i\in\mathbb{I}} := \sum_{i\in\mathbb{I}} a_i C f_i = C\sum_{i\in\mathbb{I}} a_i f_i = CT^*_F\{a_i\}_{i\in\mathbb{I}}.$$

Hence, $T^*_{CF} = CT^*_F$ and so $C^{-1}T^*_{CF} = T^*_F$. Since T^*_F is the composition of two bounded operators, so it is also a bounded operator. Now, by [8], it is concluded that $F = \{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence in $\mathcal{H}$.

In the following, we introduce the concept of (C, C')-controlled Riesz bases.

Definition 2.3. A C^2-controlled Riesz basis for a Hilbert space $\mathcal{H}$ is a family of the form $\{C^{-1}Ve_i\}_{i\in\mathbb{I}}$, where $\{e_i\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}$ and $V : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator.

Obviously, every C^2-controlled Riesz basis is a Riesz basis for $\mathcal{H}$. Analogous to Theorem 3.6.6. of [8], the following proposition gives an equivalent condition for a sequence $\{f_i\}_{i\in\mathbb{I}}$ being a C^2-controlled Riesz basis.

Proposition 2.4. A sequence $F = \{f_i\}_{i\in\mathbb{I}}$ in a Hilbert space $\mathcal{H}$ is a C^2-controlled Riesz basis if and only if F is complete and there exist constants $0 < A \leq B < \infty$ such that for every $\{a_i\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})$,

$$A\sum_{i\in\mathbb{I}} |a_i|^2 \leq \left\|\sum_{i\in\mathbb{I}} a_i C f_i\right\|^2 \leq B\sum_{i\in\mathbb{I}} |a_i|^2. \tag{2.3}$$

Proof . Let $F = \{f_i\}_{i\in\mathbb{I}}$ be a complete sequence in $\mathcal{H}$ which satisfies in (2.3). Define the operator $V : \mathcal{H} \to \mathcal{H}$ as

$$Vf := \sum_{i\in\mathbb{I}} \langle f, e_i\rangle C f_i, \qquad (f \in \mathcal{H}). \tag{2.4}$$

Then, considering $\{\langle f, e_i\rangle\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})$, we have

$$A\|f\|^2 = A\sum_{i\in\mathbb{I}} |\langle f, e_i\rangle|^2 \leq \left\|\sum_{i\in\mathbb{I}} \langle f, e_i\rangle C f_i\right\|^2 = \|Vf\|^2 \leq B\sum_{i\in\mathbb{I}} |\langle f, e_i\rangle|^2 = B\|f\|^2.$$

Therefore, V is a bounded, injective and closed range operator. Moreover, the adjoint operator

$$V^* f = \sum_{i\in\mathbb{I}} \langle f, C f_i\rangle e_i,$$

is also injective, because if $V^* f = 0$, for some $f \in \mathcal{H}$, then

$$0 = \sum_{i\in\mathbb{I}} \langle f, C f_i\rangle e_i.$$

Since $\{e_i\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}$, it implies that $\langle f, C f_i\rangle = \langle C^* f, f_i\rangle = 0$. Now, by the completeness of $F = \{f_i\}_{i\in\mathbb{I}}$ and invertibility of C, we conclude that $f = 0$. Hence, V is invertible. Furthermore,

$$C^{-1} V e_j = C^{-1} \sum_{i\in\mathbb{I}} \langle e_j, e_i\rangle C f_i = f_j.$$

Conversely, assume that there exists an invertible operator $V \in \mathcal{B}(\mathcal{H})$ such that $f_i = C^{-1} V e_i, i \in \mathbb{I}$. We show that the sequence $\{f_i\}_{i\in\mathbb{I}}$ is complete and satisfies in (2.3). For each $\{a_i\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})$,

$$\left\|\sum_{i\in\mathbb{I}} a_i C f_i\right\|^2 = \left\|\sum_{i\in\mathbb{I}} a_i V e_i\right\|^2 \leq \|V\|^2 \sum_{i\in\mathbb{I}} |a_i|^2.$$

For the lower inequality,

$$\sum_{i\in\mathbb{I}} |a_i|^2 = \left\|\sum_{i\in\mathbb{I}} a_i e_i\right\|^2 = \left\|V^{-1}V\left(\sum_{i\in\mathbb{I}} a_i e_i\right)\right\|^2 \leq \|V^{-1}\|^2 \left\|\sum_{i\in\mathbb{I}} a_i V e_i\right\|^2$$
$$= \|V^{-1}\|^2 \left\|\sum_{i\in\mathbb{I}} a_i C f_i\right\|^2 .$$

Finally, it is enough to show that $\{f_i\}_{i\in\mathbb{I}}$ is complete. Suppose that there exists $f \in \mathcal{H}$ such that for every $i \in \mathbb{I}$,

$$0 = \langle f, f_i\rangle = \langle f, C^{-1}Ve_i\rangle = \left\langle (C^{-1}V)^* f, e_i\right\rangle .$$

Since $\{e_i\}_{i\in\mathbb{I}}$ is complete, so $(C^{-1}V)^* f = 0$. Now, the invertibility of $C^{-1}V$ implies that $f = 0$. □

In the next result, we prove that every controlled Riesz basis is a controlled frame.

Theorem 2.5. In a Hilbert space $\mathcal{H}$, every C^2-controlled Riesz basis is a C^2-controlled frame.

Proof . Suppose that $\{f_i\}_{i\in\mathbb{I}}$ is a C^2-controlled Riesz basis for $\mathcal{H}$. Then, for every $f \in \mathcal{H}$,

$$\sum_{i\in\mathbb{I}} |\langle f, Cf_i\rangle|^2 = \sum_{i\in\mathbb{I}} |\langle f, Ve_i\rangle|^2 = \sum_{i\in\mathbb{I}} |\langle V^* f, e_i\rangle|^2 = \|V^* f\|^2 \leq \|V^*\|^2 \|f\|^2.$$

So $\{f_i\}_{i\in\mathbb{I}}$ is a C^2-controlled Bessel sequence. For the lower bound,

$$\|f\|^2 = \|(V^*)^{-1}V^* f\|^2 \leq \|(V^*)^{-1}\|^2 \|V^* f\|^2 = \|V^{-1}\|^2 \sum_{i\in\mathbb{I}} |\langle f, Cf_i\rangle|^2 .$$

□

One of the essential applications of frames is that they provide basis-like but generally non-unique decompositions for the elements of $\mathcal{H}$. In these decompositions, dual frames play a key role. Thus it is natural to extend the duality from frames to the case of controlled frames and examine its properties.

Definition 2.6. Suppose that $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ are two C^2 and C'^2- controlled Bessel sequences in $\mathcal{H}$. Then, G is called a (C, C')-dual of F if $T^*_{C'G} T_{CF} = I$, in other words for every $f \in \mathcal{H}$,

$$f = \sum_{i\in\mathbb{I}} \langle f, Cf_i\rangle C' g_i.$$

Every C^2-controlled frame possess at least one C^2-dual, more precisely,

$$f = \sum_{i\in\mathbb{I}} \left\langle f, C\left(C^{-1}S^{-1}_{CF}C\right) f_i\right\rangle Cf_i. \tag{2.5}$$

The sequence $\{\widetilde{f}_i\}_{i\in\mathbb{I}} = \left\{\left(C^{-1}S_{CF}^{-1}C\right) f_i\right\}_{i\in\mathbb{I}}$ is called the canonical C^2-dual of C^2-controlled frame F. Clearly, the equation (2.5) implies that

$$T_{CF}^* T_{C\tilde{F}} = I.$$

Remark 2.7. Let $F = \{f_i\}_{i\in\mathbb{I}} = \{C^{-1}Ve_i\}_{i\in\mathbb{I}}$ be a C^2-controlled Riesz basis for $\mathcal{H}$. Then, the canonical dual controlled frame

$$\widetilde{F} = \{(C^{-1}S_{CF}^{-1}C)f_i\}_{i\in\mathbb{I}} = \{(S_{CF}C)^{-1}Ve_i\}_{i\in\mathbb{I}},$$

is a (C, S_{CF})-controlled Riesz basis for $\mathcal{H}$. Furthermore, $T_{CF}T_{C\tilde{F}}^* = I$, since for every $\{a_i\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})$,

$$\begin{aligned}
T_{CF}T_{C\widetilde{F}}^*\{a_i\}_{i\in\mathbb{I}} = \left\{\left\langle \sum_{i\in\mathbb{I}} a_i C\widetilde{f}_i, Cf_j \right\rangle\right\}_{j\in\mathbb{I}} &= \left\{\sum_{i\in\mathbb{I}} a_i \left\langle V^* S_{CF}^{-1} V e_i, e_j\right\rangle\right\}_{j\in\mathbb{I}} \\
&= \left\{\sum_{i\in\mathbb{I}} a_i \left\langle V^*(VV^*)^{-1} V e_i, e_j\right\rangle\right\}_{j\in\mathbb{I}} \\
&= \left\{\sum_{i\in\mathbb{I}} a_i \delta_{ij}\right\}_{j\in\mathbb{I}} \\
&= \{a_i\}_{i\in\mathbb{I}}.
\end{aligned}$$

Now, we remember the concept of multipliers for controlled Bessel sequences, which is defined, of course by one controller operator, in [11].

Definition 2.8. Let $m \in \ell^\infty(\mathbb{I})$ and $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ be two C^2 and C'^2 controlled Bessel sequences in $\mathcal{H}$, with bounds B and B', respectively. Then, the operator $M_{m,(C'G),(CF)} : \mathcal{H} \to \mathcal{H}$ defined as

$$M_{m,(C'G),(CF)}f := \sum_{i\in\mathbb{I}} m_i \langle f, Cf_i\rangle C' g_i, \quad (f \in \mathcal{H}),$$

is called the (C, C')-controlled Bessel multiplier operator with symbol m. It is clear that

$$M_{m,(C'G),(CF)} = T_{C'G}^* \mathfrak{M}_m T_{CF},$$

where $\mathfrak{M}_m : \ell^2(\mathbb{I}) \to \ell^2(\mathbb{I})$ is the mapping given by

$$\mathfrak{M}_m\{a_i\} := m_i a_i, \quad (\{a_i\}_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})).$$

Furthermore, the operator $M = M_{m,(C'G),(CF)}$ is bounded and adjointable with $\|M\| \leq \sqrt{BB'}\|m\|_\infty$ and $M^* = M_{\overline{m},(CF),(C'G)}$.

3 Main Results

In this section, we introduce the concept of generalized controlled Bessel multipliers and investigate some properties of them.

Definition 3.1. Let $U \in \mathcal{B}(\ell^2(\mathbb{I}))$ and $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ be two C^2 and C'^2-controlled Bessel sequences in $\mathcal{H}$, respectively. Then, the operator $M_{(C'G)U(CF)} : \mathcal{H} \to \mathcal{H}$ defined as $M_{(C'G)U(CF)} = T^*_{C'G}UT_{CF}$ is called the generalized (C, C')-controlled Bessel multiplier.

In the first proposition, we summarize some properties of $M_{(C'G)U(CF)}$.

Proposition 3.2. Let $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ be two C^2 and C'^2-controlled Bessel sequences in $\mathcal{H}$ with upper bounds B and B', respectively, and $M_{(C'G)U(CF)}$ be the generalized Bessel multiplier associated with F and G. Then,

(1) $M_{(C'G)U(CF)}$ is a bounded and linear operator with $\|M_{(C'G)U(CF)}\| \leq \sqrt{BB'}\|U\|$.

(2) If $U \in \mathcal{S}_p(\ell^2(\mathbb{I}))$, then $M_{(C'G)U(CF)} \in \mathcal{S}_p(\mathcal{H})$. The converse is true only if F and G are C^2 and C'^2-controlled Riesz bases, respectively.

Proof . (1)

$$\|M_{(C'G)U(CF)}\| = \|T^*_{C'G}UT_{CF}\| \leq \|T^*_{C'G}\|\|U\|\|T_{CF}\| \leq \sqrt{BB'}\|U\|.$$

(2) Let $U \in \mathcal{S}_p(\mathcal{H})$. By the fact that $\mathcal{S}_p(\mathcal{H})$ is a two sided ideal of $\mathcal{B}(\mathcal{H})$ and $M_{(C'G)U(CF)} = T^*_{C'G}UT_{CF}$, the result is obtained. For the second part, suppose that $M_{(C'G)U(CF)} \in \mathcal{S}_p(\mathcal{H})$ and $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ are C^2 and C'^2-controlled Riesz bases in $\mathcal{H}$. Then, the operators T_{CF} and $T^*_{C'G}$ are surjective and injective, respectively. Therefore, there are a right inverse operator R for T_{CF} and a left inverse operator L for $T^*_{C'G}$ such that

$$T_{CF}R = I, \qquad LT^*_{C'G} = I.$$

Hence, we can write

$$U = LM_{(C'G)U(CF)}R. \tag{3.1}$$

Now, by the fact that $\mathcal{S}_p(\mathcal{H})$ is a two sided ideal of $\mathcal{B}(\mathcal{H})$, the result is obtained. □ The following proposition provides some conditions under which the underlying controlled Bessel sequences of a generalized (C, C')-controlled Bessel multiplier become controlled frames.

Proposition 3.3. Assume that $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ are two C^2 and C'^2-controlled Bessel sequences in $\mathcal{H}$. Moreover, suppose that there exists $\lambda > 0$ such that for every $f \in \mathcal{H}$, $\lambda\|f\|^2 \leq \left|\langle M_{(C'G)U(CF)}f, f\rangle\right|$. Then, F and G are C^2 and C'^2-controlled frames in $\mathcal{H}$, respectively.

Proof . Let $F=\{f_i\}_{i\in\mathbb{I}}$ and $G=\{g_i\}_{i\in\mathbb{I}}$ be two C^2 and C'^2-controlled Bessel sequences in $\mathcal{H}$ with bounds B and B' ,respectively. We denote $M_{(C'G)U(CF)}$ by M. By the assumption, there exists $\lambda>0$ such that for every $f\in\mathcal{H}$, $\lambda\|f\|^2\leq|\langle Mf,f\rangle|$. Then,

$$\lambda\|f\|^2\leq|\langle Mf,f\rangle|\leq\|Mf\|\|f\|.$$

So, M is a bounded below operator. By the same argument, we can show that M^* is also bounded below. Now, we show that F is a C^2-controlled frame. Since M is bounded below, then for every $f\in\mathcal{H}$ we can find some $g\in\mathcal{H}$ such that $\|g\|=1$ and $\lambda\|f\|\leq|\langle Mf,g\rangle|$. So

$$\begin{aligned}\lambda\|f\|\leq|\langle Mf,g\rangle|&=|\langle T^*_{C'G}UT_{CF}f,g\rangle|\\&=|\langle T_{CF}f,U^*T_{C'G}g\rangle|\\&\leq\left(\sum_{i\in\mathbb{I}}|\langle f,Cf_i\rangle|^2\right)^{\frac{1}{2}}\|U^*\|\sqrt{B'}.\end{aligned}$$

Hence, for every $f\in\mathcal{H}$,

$$\lambda^2B'^{-1}\|U^*\|^{-2}\|f\|^2\leq\sum_{i\in\mathbb{I}}|\langle f,Cf_i\rangle|^2.$$

Therefore, the sequence F is a C^2-controlled frame. By a similar argument and using the fact that M^* is also a bounded below operator, one can show that the sequence G is a C'^2-controlled frame. □

Corollary 3.4. Assume that $F=\{f_i\}_{i\in\mathbb{I}}$ and $G=\{g_i\}_{i\in\mathbb{I}}$ are two C^2 and C'^2-controlled Bessel sequences in $\mathcal{H}$. If $M_{(C'G)U(CF)}$ is invertible, then F and G are C^2 and C'^2-controlled frames in $\mathcal{H}$, respectively.

Proof . Since $M_{(C'G)U(CF)}$ is invertible, then both $M_{(C'G)U(CF)}$ and $M^*_{(C'G)U(CF)}$ are bounded below operators and so the result is obtained from Proposition 3.3. □ The next result gives a necessary and sufficient condition for the invertibility of the generalized Riesz multiplier.

Proposition 3.5. Let $F=\{f_i\}_{i\in\mathbb{I}}$ and $G=\{g_i\}_{i\in\mathbb{I}}$ be two C^2 and C'^2-controlled Riesz bases in $\mathcal{H}$. Then, the multiplier $M_{(C'G)U(CF)}$ is invertible if and only if U is invertible.

Proof . Assume that $U\in\mathcal{B}(\ell^2(\mathbb{I}))$ is an invertible operator. Put $M^{-1}_{(C'G)U(CF)}=M_{(C\widetilde{F})U^{-1}(C'\widetilde{G})}$, where $\widetilde{F}$ and $\widetilde{G}$ are the canonical controlled duals of F and G. Then, by Remark 2.7, we have

$$M_{(C'G)U(CF)}M_{(C\widetilde{F})U^{-1}(C'\widetilde{G})}=(T^*_{C'G}UT_{CF})(T^*_{C\widetilde{F}}U^{-1}T_{C'\widetilde{G}})=I.$$

Similarly, we can show that $M_{(C\tilde{F})U^{-1}(C'\tilde{G})}M_{(C'G)U(CF)} = I$. Conversely, suppose that $M_{(C'G)U(CF)}$ is invertible with the inverse M^{-1}. Then

$$U\left(T_{CF}M^{-1}T^*_{C'G}\right) = T_{C'\widetilde{G}}T^*_{C'G}\left(UT_{CF}M^{-1}\right)T^*_{C'G} = T_{C'\widetilde{G}}T^*_{C'G} = I.$$

A similar argument shows that $\left(T_{CF}M^{-1}T^*_{C'G}\right)U = I$. □

As we have seen in Proposition 3.5, if $U \in \mathcal{B}(\ell^2(\mathbb{I}))$ is invertible, the generalized Riesz multiplier $M_{(C'G)U(CF)}$ is automatically invertible and vise versa. Moreover, $M^{-1}_{(C'G)U(CF)} = M_{(C\widetilde{F})U^{-1}(C'\widetilde{G})}$. This result motivates us to generalize this idea for controlled frames. In more details, the following proposition shows that there are other invertible frame multipliers $M_{(C'G)U(CF)}$ whose inverses can be represented as multipliers using the inverted symbol and suitable dual frames of F and G.

Proposition 3.6. Let $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ be two C^2 and C'^2-controlled frames in $\mathcal{H}$ and U be an invertible operator in $\mathcal{B}(\ell^2(\mathbb{I}))$. Assume that the generalized multiplier $M_{(C'G)U(CF)}$ is invertible. If $M^{-1}_{(C'G)U(CF)}$ commutes with C, then there exists a dual frame $G^\dagger$ of G such that for any dual frame F^d of F, we have $M^{-1}_{(C'G)U(CF)} = M_{(C'F^d)U^{-1}(CG^\dagger)}$.

Proof . We denote $M_{(C'G)U(CF)}$ by M. Let $\{\delta_i\}_{i\in\mathbb{I}}$ be the standard orthonormal basis for $\ell^2(\mathbb{I})$. Then, the sequence $G^\dagger = \{(UT_FM^{-1})^*\delta_i\}_{i\in\mathbb{I}}$ is a (C, C')-dual frame of G, since

$$\begin{aligned}\sum_{i\in\mathbb{I}}\left\langle f, C(UT_FM^{-1})^*\delta_i\right\rangle C'g_i &= \sum_{i\in\mathbb{I}}\left\langle f, (M^{-1})^*CT^*_FU^*\delta_i\right\rangle C'g_i\\ &= \sum_{i\in\mathbb{I}}\left\langle UT_{CF}M^{-1}f, \delta_i\right\rangle C'g_i\\ &= T^*_{C'G}\left(UT_{CF}M^{-1}\right)f\\ &= MM^{-1}f = f.\end{aligned}$$

On the other hand,

$$\begin{aligned}(M^{-1})^*T^*_{CF}U^*\delta_i &= (M^{-1})^*CT^*_FU^*\delta_i\\ &= C(M^{-1})^*T^*_FU^*\delta_i\\ &= C(UT_FM^{-1})^*\delta_i\\ &= T^*_{CG^\dagger}(U^{-1})^*(U^*\delta_i).\end{aligned}$$

Since U is surjective, so we have

$$(M^{-1})^*T^*_{CF} = T^*_{CG^\dagger}(U^{-1})^*.$$

Now, for every (C, C')-dual frame F^d of F, we have

$$(M^{-1})^* = T^*_{CG^\dagger}(U^{-1})^*T_{C'F^d},$$

and therefore

$$M^{-1} = T^*_{C'F^d}(U^{-1})T_{CG^\dagger} = M_{(C'F^d)U^{-1}(CG^\dagger)}.$$

□

4 Acknowledgement

The author would like to thank the referees for their valuable comments and suggestions which improved the manuscript.

References

[1] Gh. Abbaspour Tabadkan, H. Hosseinnezhad, and A. Rahimi, *Generalized Bessel multipliers in Hilbert spaces*, Result. Math., **73** 18p. (2018). zbl MR doi

[2] M. L. Arias, M. Pacheco, *Bessel fusion multipliers*, J. Math. Anal. Appl., **348** (2008) 581–588. zbl MR doi

[3] P. Balazs, *Basic definition and properties of Bessel multipliers*, J. Math. Anal. Appl., **325** (2007) 571–585. zbl MR doi

[4] P. Balazs, *Hilbert-Schmidt operators and frames classification, approximation by multipliers and algorithms*, Int. J. Wavelets Multiresolut. Inf. Process., **6** (2008) 315–330. zbl MR doi

[5] P. Balazs, J. P. Antoine, and A. Gryboś, *Weighted and controlled frames: Mutual relationship and first numerical properties*, Int. J. Wavelets Multiresolut. Inf. Process., **8** (2010) 109–132. zbl MR doi

[6] P. Balazs, B. Laback, G. Eckel, and W. A. Deutsch, *Time-frequency sparsity by removing perceptually irrelevant components using a simple model of simultaneous masking*, IEEE Trans. Audio Speech Language Process., **18** (2010) 34–49. doi

[7] P. Balazs, D. Bayer, and A. Rahimi, *Multipliers for continuous frames in Hilbert spaces*, J. Phys. A, Math. Theor., **45** (2012) 20. zbl MR doi

[8] O. Christensen, *An Introduction to Frames and Riesz Bases*, 2nd ed., Basel: Birkhäuser, 2016. zbl MR doi

[9] R. J. Duffin, A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Am. Math. Soc., **72** (1952) 341–366. zbl MR doi

[10] A. Khosravi, M. Mirzaee Azandaryani, *Bessel multipliers in Hilbert C^*-modules*, Banach J. Math. Anal., **9** (2015) 153–163. zbl MR doi

[11] M. R. Kouchi, A. Rahimi, and F. A. Shah, *Duals and multipliers of controlled frames in Hilbert spaces*, Int. J. Wavelets Multiresolut. Inf. Process., **16** 13p. (2018). zbl MR doi

[12] P. Majdak, P. Balazs, W. Kreuzer, and M. Dörfler, *A time-frequency method for increasing the signal-to-noise ratio in system identification with exponential sweeps*, Proceedings of the 36th International Conference on Acoustics, Speech, and Signal Processing, Prague Congress Center, Prague, Czech Republic, 2011. doi

[13] A. Rahimi, *Multipliers of generalized frames in Hilbert spaces*, Bull. Iran. Math. Soc., **37** (2011) 63–80. zbl MR

[14] A. Rahimi, P. Balazs, *Multipliers for p-Bessel sequences in Banach spaces*, Integr. Equ. Oper. Theory, **68** (2010) 193–205. zbl MR doi

[15] A. Rahimi, A. Fereydooni, *Controlled g-frames and their g-multipliers in Hilbert spaces*, Analele Stiint. ale Univ. Ovidius Constanta Ser. Mat., **21** (2013) 223–236. zbl MR doi

[16] R. Schatten, *Norm Ideals of Completely Continuous Operators*, Springer-Verlag, Berlin, 1970. zbl MR

[17] D. Wang, G. J. Brown, *Computational Auditory Scene Analysis: Principles, Algorithms, and Applications*, Wiley-IEEE Press, 2006. link