



A Note on Normal Generalized Topological Spaces

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Abstract

In this paper, we show by means of examples that the Urysohn's lemma is not necessarily valid for a normal generalized topological space. Using a GT-analog of the Urysohn's lemma, we show that the following two statements are equivalent for a Hausdorff normal generalized topological space X with the property that each closed set is a G_δ -set:

- (i) X is a topological space.
- (ii) The set of nonnegative lower semi-continuous functions on X is closed under the operations of addition and multiplication.

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1 Introduction

The theory of generalized topological space was founded by Á. Császár [2]. Let X be a non-empty set, following [2] a *generalized topology* (briefly GT) on X is a collection μ of subsets of X which is closed under union and contains the empty set \emptyset . The members of μ are called μ -open sets and their complements are called μ -closed sets. There are many valuable papers which study properties of generalized topological spaces (see for examples [6, 5] and references therein).

A natural question in the theory of GT spaces is, under what conditions a GT space is a topological space. In [1], we show that a generalized topological space has net-closure property or net-continuous property if and only if it is a topological space. In this paper we answer to this question for certain normal GT spaces.

A GT space is called normal if one can separate disjoint closed sets by disjoint open sets. We say a function f from (X, μ) to (X', μ') is (μ, μ') -continuous (briefly continuous) if $f^{-1}(U) \in \mu$ for all $U \in \mu'$. The Urysohn's lemma for a usual normal topological space X , see [4], says that for every two disjoint closed subsets A and B , there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$ (consider $[0, 1]$ with the usual topology). As we shall see in Example 2.2, this is not true, in general, for a GT space. In [3], Á. Császár proved an Urysohn type lemma for a GT space, however he put a special GT structure on $[0, 1]$. As we mention in Lemma 2.4, if we consider $[0, 1]$ with the usual topology, then lower-semi-continuous functions play the role of continuous functions in the standard Urysohn's lemma for usual topological space. On the other hand, we know that for a usual topological space X , $C_l^+(X)$, the set of all nonnegative lower semi-continuous functions on X , is closed under addition and multiplication. We will see in Remark 2.9 and Example 2.10 that this is not true for a GT space in general. This situation leads us to the following question :

Question: Is a GT space X a topological space, when $C_l^+(X)$ is closed under addition and multiplication?

We will show that the answer to the above question is affirmative if X is normal and each closed set is a G_δ -set.

2 Main results

To state and prove of the main result of this paper, Theorem 2.7, we need some definitions and lemmas.

Definition 2.1. Let (X, μ) be a GT space, we say that X is a normal GT space if for each

two disjoint μ -closed subsets A and B of X , there are two disjoint μ -open sets U and V which contains A and B , respectively.

Let (X, μ) be a GT space by proposition 2.2 of [3], X is normal if and only if for each μ -closed set A and μ -open set U containing A , there is a μ -open set V such that $A \subset V \subset \overline{V} \subset U$. For a normal topological space X the Urysohn's lemma says that for every two disjoint closed set A and B there is a real continuous function f from X to $[0, 1]$ with $f(A) = 0$ and $f(B) = 1$.

The following example shows that this lemma is not necessarily valid for GT spaces.

Example 2.2. Let \mathbb{R} be the real line and let μ be the set of all subsets U of \mathbb{R} which U is open in the standard topology of \mathbb{R} and $U \cap \mathbb{N} \neq \emptyset$. It is clear that (\mathbb{R}, μ) is a GT space which is not a topological space. We prove that μ is a Hausdorff normal GT on \mathbb{R} , but the Urysohn's lemma fails. Let A and B be two disjoint μ -closed subsets of \mathbb{R} . Then there are natural numbers m and n , $m \neq n$ such that $m \notin A$ and $n \notin B$. Put $\tilde{A} = A \cup \{n\}$ and $\tilde{B} = B \cup \{m\}$. \tilde{A} and \tilde{B} are two disjoint closed subsets of \mathbb{R} with the standard topology. Since \mathbb{R} with the standard topology is normal, there are two disjoint standard open sets U and V containing \tilde{A} and \tilde{B} , respectively. U and V are μ -open sets and separate A and B . So (\mathbb{R}, μ) is a normal GT space. Consider two disjoint μ -closed sets $A = (-\infty, 1]$ and $B = [3/2, +\infty)$. We claim that there is no continuous function $f : (\mathbb{R}, \mu) \rightarrow [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. Otherwise $f^{-1}(0, 1)$ must be μ -open since f is μ -continuous, but $f^{-1}(0, 1) \subset (1, 3/2)$ have empty intersection with natural numbers. On the other hand f is continuous in the standard topology of \mathbb{R} , because every μ -open set is open in standard topology hence intermediate value theorem implies that $f^{-1}(0, 1)$ is not empty, this is a contradiction.

Definition 2.3. Let (X, μ) be a GT space, a function $f : X \rightarrow \mathbb{R}$ is called lower semi-continuous at x_0 if for all $\epsilon > 0$, there exists μ -open set U containing x_0 such that

$$f(x) > f(x_0) - \epsilon, \quad \forall x \in U.$$

We say f is lower semi-continuous on X if f is lower semi-continuous at each $x \in X$. We denote by $C_l(X)$, the set of all lower semi-continuous functions on X . $C_l^+(X)$ consists of all non-negative elements of $C_l(X)$.

Remark 2.4. The following are immediate consequence of definitions:

- (i) $f \in C_l(X)$ if and only if $f^{-1}(a, +\infty) \in \mu$ for all $a \in \mathbb{R}$.
- (ii) The uniform limit of lower semi-continuous functions is lower semi-continuous.
- (iii) If m is the minimum value for $f \in C_l(X)$, then $f^{-1}\{m\}$ is μ -closed since $f^{-1}(m, +\infty)$ is μ -open.

A GT-analogy for the standard Urysohn's lemma, can be stated as follows which proof is a consequence of Theorem 3.3 in [3] and the above remark part (i).

Lemma 2.5. Let (X, μ) be a Hausdorff normal GT space, and A, B be two disjoint μ -closed sets, then there exists an $f \in C_l(X)$ such that for all $x \in X$, $0 \leq f(x) \leq 1$,

$$f(A) = 0, \quad f(B) = 1.$$

Definition 2.6. We say a subset A of X is a G_δ set if $A = \bigcap_{i=1}^{\infty} G_i$ where G_i 's are open sets.

Lemma 2.7. Let X be a Hausdorff and normal GT space and A be a closed set which is a G_δ set, further assume that $C_l^+(X)$ is closed under addition. Then there is an $f : X \rightarrow [0, 1]$, $f \in C_l^+(X)$, such that $A = f^{-1}\{0\}$.

Proof . Let $A = \bigcap_{i=1}^{\infty} G_i$, then A and G_i^c are two disjoint closed sets. Then there is an $f_i \in C_l^+(X)$, $f_i(x) \leq 1/2^i$ such that $f_i(A) = 0$ and $f_i(G_i^c) = 1/2^i$. Now for $f = \sum_{i=1}^{\infty} f_i$ we have $f \in C_l^+(X)$ and $A = f^{-1}\{0\}$. \square

Theorem 2.8. Let (X, μ) be a Hausdorff and normal GT space such that $X \in \mu$ and each μ -closed set is a G_δ -set. If $C_l^+(X)$ is closed under addition and multiplication, then (X, μ) is a topological space.

Proof . To prove that (X, μ) is a topological space, it is enough to show that $A \cup B$ is μ -closed for any two μ -closed sets A and B . By the above lemma there are two functions $f, g : X \rightarrow [0, 1]$ such that $f, g \in C_l(X)$, $A = f^{-1}\{0\}$ and $B = g^{-1}\{0\}$. Since $C_l^+(X)$ is closed under multiplication, then $fg \in C_l^+(X)$ and we have $(fg)^{-1}\{0\} = A \cup B$. Thus $A \cup B$ is μ -closed, by Remark 2.1 part (iii), and this completes the proof. \square

Remark 2.9. In the above theorem, we can not drop the condition of closedness of $C_l^+(X)$ under addition and multiplication. To see this, let X be the same space as in Example 2.2, we proved there, that X is a normal GT space for which the Urysohn's lemma fails. Now we show that each closed set in X is a G_δ set. Let A be a μ -closed set. Then A is a closed subset of real line with the standard topology. It is well known that every closed set in a metric space is a G_δ set. Then there are standard open sets G_i with $A = \bigcap_{i=1}^{\infty} G_i$. Now put $\tilde{G}_i = G_i \cup \{i\}$. Then \tilde{G}_i 's are μ open sets and $A = \bigcap_{i=1}^{\infty} \tilde{G}_i$.

Example 2.10. Let $X = \{a, b, c\}$, $\mu = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$. It can be easily proved that (X, μ) is a normal and Hausdorff GT space, each closed set is a G_δ -set, but X is not a topological space. Then Theorem 2.8 shows that $C_l^+(X)$ is not closed under addition and multiplication, simultaneously.

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