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Classical prime injective S-acts

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Abstract

In this article, we define classical prime right ideals of semigroups and generalize them to classical prime subacts of acts. We study the notion of $\mathcal{M}$-injectivity where $\mathcal{M}$ is the class of all classical prime monomorphisms. We investigate the Skornjakov criterion with respect to classical prime injectivity of acts. We characterize their behavior under well-known constructions such as the product, coproduct and direct sum. Among other results, it is proved that an S-act is classical prime injective if and only if it is a classical prime-absolute retract, if and only if it has no classical prime-essential extension.

Keywords: Classical prime ideal, Classical prime subact, Classical prime injective.
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1 Introduction and Preliminaries

Let S be a monoid with 1 as its identity. A (right) *unitary S-act* A is a set A and a function $A \times S \to A$ such that if as denotes the image of (a, s) for $a \in A$ and $s \in S$, then

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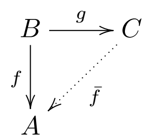
(i) $(as)t = a(st)$ for all $a \in A$, $s, t \in S$; and (ii) $a1 = a$ for all $a \in A$. A *homomorphism* $f : A \to B$ between S-acts A and B is a function with $f(as) = f(a)s$ for each $a \in A$, $s \in S$. We denote the category of all S-acts and homomorphisms between them by **Act-S**. An S-subact B of an S-act A, written as $B \leq A$, is a subset B of A such that $bs \in B$ for all $b \in B$ and $s \in S$. Thus subacts of the S-act S are precisely right ideals of S. Note that S is naturally a unitary S-act under its own monoid action (i.e., the action is given by multiplication in S). In the present text, we have occasion to consider the word ideal, two-sided ideal. An element θ in an S-act A with $\theta s = \theta$, for all $s \in S$, is called a *fixed or zero element* of A, and $\Theta = \{\theta\}$ is a subact of A.

The concept of the prime ideal, which arises in the theory of rings as a generalization of the concept of prime number in the ring of integers, plays a highly important role in that theory, and prime ideals are also useful tools in the theory of semigroups, too; see [7]. Recall that an ideal I of S is called *prime* if for $s, s' \in S$, the inclusion $sSs' \subseteq I$ implies that either $s \in I$ or $s' \in I$; see [7]. Equivalently, I is prime if and only if for any (right and left) ideals J and K of S, the set inclusion $JK \subseteq I$ implies $J \subseteq I$ or $K \subseteq I$ (see [10, Proposition 4.10.2]). This notion was extended to arbitrary S-act, analogous to the notion of prime modules have been introduced by Dauns in [8]. Let B be a subact of an act A. The set $(B : A) = \{s \in S : As \subseteq B\}$ is a right ideal of S, which is called the *associated right ideal* of B. A subact B of an S-act A is a *prime* subact if for any $a \in A$ and $s \in S$, the inclusion $aSs \subseteq B$ implies either $a \in B$ or $s \in (B : A)$. A right ideal I of S is prime if and only if I is prime as a subact of S. An S-act A itself is called *prime* if the subact $\Theta = \{\theta\}$ of A is prime as a subact of A ; see [1].

In this paper, we define some new notions such as classical prime right ideals in semigroups, classical prime subacts of acts, and classical prime homomorphisms between acts. We indicate the relation between a classical prime right ideal and a classical prime subact of an act and its associated right ideal.

We call a monomorphism $f : B \to A$, as an embedding of B into A. If such a monomorphism exists, we say that B can be embedded into A, that A contains an isomorphic copy of B, or that A is an extension of B.

Banaschewski indicated the notion of $\mathcal{M}$-injectivity in a category $\mathcal{A}$, where $\mathcal{M}$ is a subclass of monomorphisms that the members of which may be called $\mathcal{M}$-morphisms, and it is defined as follow: an S-act A is said to be $\mathcal{M}$-injective if for any $\mathcal{M}$-morphism $g : B \to C$, any morphism $f : B \to A$ can be lifted to a morphism $\bar{f} : C \to A$, such that $\bar{f}g = f$ [2].



Eventually, we are going to study $\mathcal{M}$-injectivity of acts, where $\mathcal{M}$ is the class of all classical prime monomorphisms. We know that the Skornjakov–Baer criterion for injectivity is true for the category of **Act-S**. The results of the current study indicate that classical prime injectivity of acts which contain a fixed element is equivalent to being injective relative to all inclusion of classical prime subacts of cyclic acts. In Proposition 2.18, we show that all kinds of classical prime injectivity of acts (with zero) are preserved by taking products (direct sums), and if a product (direct sum) is of a kind of classical prime injectivity, then all its components are of a kind of classical prime injectivity. But this characteristic is valid for coproducts of S-acts over special kind of monoids (see Proposition 2.21).

One of the main results of this paper is Banaschewski's theorem for classical prime injective acts, which presents the equivalent conditions of classical prime injectivity of acts (see Theorem 2.26).

We express our definition that is used to this article. We call that an act A is *decomposable* if there exist two proper subacts B and C of A such that $A = B \cup C$ and $B \cap C = \varnothing$. In this case, $A = B \cup C$ is called a *decomposition* of A.

A monomorphism $h : A \to B$ is called *essential* if any homomorphism $g : B \to C$ is a monomorphism whenever gh is a monomorphism.

An extension B of an act A is called *injective hull* of A if it is an essential extension of A, which is also injective. It will be indicated that for any act, there exists a classical prime injective hull (see Corollary 2.27).

2 Classical prime injective acts

In [4], classical prime submodules have been studied under the name"weakly prime submodule". In this section, we define two notions, which are generalizations of classical prime right ideals and classical prime subact of an act, analogous to their counterparts in module theory. We then investigate some of their key properties.

Definition 2.1. (1) A right ideal I of a semigroup S is called a *classical prime right ideal*, whenever the inclusion $sSs'Ss'' \subseteq I$, for any $s, s', s'' \in S$, implies that $sSs' \subseteq I$ or $sSs'' \subseteq I$.

(2) A subact B of an act A is called a *classical prime subact*, whenever the inclusion $aSs'Ss'' \subseteq B$ for any $s', s'' \in S$ and $a \in A$, implies that $aSs' \subseteq B$ or $aSs'' \subseteq B$.

The semigroup S itself is always a classical prime ideal of S.
Definition 2.1 implies the following proposition obviously.

Proposition 2.2. A right ideal I of a semigroup S is classical prime if and only if I is classical prime as a subact of the S-act S.

Proposition 2.3. Every prime subact of an act is a classical prime subact.

Proof . Let B be a prime subact of A and $aS(s'Ss'') = aSs'Ss'' \subseteq B$ for any $s', s'' \in S$ and $a \in A$. Since B is a prime subact of A, we have $a \in B$ or $s'Ss'' \subseteq (B : A)$. If $a \in B$, then $aSs' \subseteq B$ or $aSs'' \subseteq B$. Now, suppose that $s'Ss'' \subseteq (B : A)$. Therefore $s' \in (B : A)$ or $s'' \in (B : A)$, since $(B : A)$ is a prime ideal of S by Proposition 3 of [1]. Hence $As' \subseteq B$ or $As'' \subseteq B$. Then we conclude that $aSs' \subseteq B$ or $aSs'' \subseteq B$, since $AS \subseteq A$ and $a \in A$. Thus B is a classical prime subact of A. □

Before we continue, let us show that a classical prime (subact) ideal needs not be a prime ideal (subact).

Example 2.4. (1) Let $(\mathbb{N}, .)$ be the natural numbers with respect to multiplication. The ideal $2\mathbb{N}$ of $\mathbb{N}$ is a classical prime ideal and it is a classical prime subact of $\mathbb{N}$. Generally, for any prime integer p, $p\mathbb{N}$ is prime and classical prime subact of $\mathbb{N}$.

Here, notice that the $\mathbb{N}$-act $3\mathbb{N} \sqcup 5\mathbb{N} = (1, 3\mathbb{N}) \cup (2, 5\mathbb{N})$ is a classical prime subact of $\mathbb{N} \sqcup \mathbb{N} = (1, \mathbb{N}) \cup (2, \mathbb{N})$, since it is a coproduct of two classical prime subacts of $\mathbb{N}$ (see Proposition 2.7(4)). But $3\mathbb{N} \sqcup 5\mathbb{N}$ is not prime subact of $\mathbb{N} \sqcup \mathbb{N}$. That is because $(2, 3)\mathbb{N}5 = (2, 15\mathbb{N}) \subseteq 3\mathbb{N} \sqcup 5\mathbb{N}$ for $(2, 3) \in \mathbb{N} \sqcup \mathbb{N}$ and $5 \in \mathbb{N}$, while $(2, 3) \notin 3\mathbb{N} \sqcup 5\mathbb{N}$ and $(\mathbb{N} \sqcup \mathbb{N})5 = 5\mathbb{N} \sqcup 5\mathbb{N} \nsubseteq 3\mathbb{N} \sqcup 5\mathbb{N}$.
(2) Take an act $A = \{a, b, c\}$ over the monoid $(\mathbb{N}, .)$, presented by the following multiplication table:

	1	2	3	4	...
a	a	a	a	a	...
b	b	b	b	b	...
c	c	b	c	b	...

that is, a and b are fixed elements and $c(2n + 1) = c$ and $c(2n) = b$ for any $n \in \mathbb{N}$. It is not difficult to check that the subact $B = \{b, c\}$ of A is classical prime and prime.
(3) Suppose that $S = \{0, s^1, s^2, \ldots, s^t\}$, $t \geq 4$, is a semigroup with multiplication $s^i s^j = s^{i+j}$ whenever $i + j \leq t$ and $s^i s^j = 0$ whenever $i + j > t$. Any proper ideal of S is in the form of $I_1 = \{0\}$ and $I_k = \{0, s^k, s^{k+1}, \ldots, s^t\}$ for $2 \leq k \leq t$ which are not prime ideals of S.

The ideals I_1 and I_k, for $4 \leq k \leq t$, are not classical prime ideals of S. For instance, the ideal I_4 is not a classical prime ideal of S, since we have $s^1Ss^1Ss^1 = \{0, s^5, \dots, s^t\} \subseteq I_4 = \{0, s^4, \dots, s^t\}$, but $s^1Ss^1 = \{0, s^3, \dots, s^t\} \nsubseteq I_4$. Also it is not difficult to check that two ideals I_2 and I_3 are classical prime whereas they are not prime.

Now we give an equivalent definition for classical prime subacts of acts over monoids.

Theorem 2.5. Let B be a subact of an act A over a monoid S. Then the following statements are equivalent:

(1) The subact B is a classical prime subact of A.

(2) For any left ideals I and J of S and for any $a \in A$, the inclusion $aIJ \subseteq B$ implies that $aI \subseteq B$ or $aJ \subseteq B$.

Proof . (1) $\Rightarrow$ (2) Let $aIJ \subseteq B$ and $aI \nsubseteq B$ for left ideals I and J of S and $a \in A$. Therefore, $aSISJ \subseteq aIJ \subseteq B$ and there exists $i \in I$ such that $ai \notin B$. It concludes that $aSiSj \subseteq B$ for any $j \in J$. It implies that $aSi \subseteq B$ or $aSj \subseteq B$. Since $ai = a1i \in aSi$, we have $aSi \nsubseteq B$. Therefore, $aSj \subseteq B$ for any $j \in J$, and then $aSJ \subseteq B$, which implies $aJ = a1J \subseteq B$.
(2) $\Rightarrow$ (1) Let $aSs'Ss'' \subseteq B$ for $s', s'' \in S$ and $a \in A$. Set left ideals $I = Ss'$ and $J = Ss''$. It implies that $aSs' \subseteq B$ or $aSs'' \subseteq B$. □

As a corollary of Theorem 2.5, we have the following.

Corollary 2.6. Let B be a subact of an act A over a monoid S. Then B is a classical prime subact of A if and only if for any $a \in A \setminus B$, $B_a = \{s \in S : as \in B\}$ is a prime right ideal of S.

Proof . Let B be a classical prime subact of A and $JK \subseteq B_a$ for left ideals J and K of S and $a \in A \setminus B$. Then $aJK \subseteq B$. It implies that $aK \subseteq B$ or $aJ \subseteq B$, by Theorem 2.5. Hence, $K \subseteq B_a$ or $J \subseteq B_a$.
Let B_a be a prime right ideal of S and $aJK \subseteq B$ for $a \in A$ and left ideals J and K of S. Then $JK \subseteq B_a$. Hence, $J \subseteq B_a$ or $K \subseteq B_a$ since B_a is a prime right ideal. Thus $aJ \subseteq B$ or $aK \subseteq B$. □

Proposition 2.7. Suppose that B_1 and B_2 are subacts of two S-acts A_1 and A_2 respectively. Then

(1) $B_1 \times A_2$ is a classical prime subact of $A_1 \times A_2$ if and only if B_1 is a classical prime subact of A_1.

(2) $A_1 \times B_2$ is a classical prime subact of $A_1 \times A_2$ if and only if B_2 is a classical prime subact of A_2.

(3) If $B_1 \times B_2$ is a classical prime subact of $A_1 \times A_2$, then B_1 is a classical prime subact of A_1 and B_2 is a classical prime subact of A_2.

(4) $B_1 \sqcup B_2$ is a classical prime subact of $A_1 \sqcup A_2$ if and only if B_1 is a classical prime subact of A_1 and B_2 is a classical prime subact of A_2.

Proof . (1) Let $B_1 \times A_2$ be a classical prime subact of $A_1 \times A_2$ and let $a_1SsSs' \subseteq B_1$ for $s, s' \in S$, $a_1 \in A_1$. Then $(a_1, a_2)SsSs' = (a_1SsSs', a_2SsSs') \subseteq B_1 \times A_2$ for any $a_2 \in A_2$. It implies that $(a_1Ss, a_2Ss) \subseteq B_1 \times A_2$ or $(a_1Ss', a_2Ss') \subseteq B_1 \times A_2$. Hence, $a_1Ss \subseteq B_1$ or $a_1Ss' \subseteq B_1$.

Now let B_1 be a classical prime subact of A_1 and let $(a_1, a_2)SsSs' = (a_1SsSs', a_2SsSs') \subseteq B_1 \times A_2$ for any $(a_1, a_2) \in A_1 \times A_2$, $s, s' \in S$. It can be concluded that $a_1SsSs' \subseteq B_1$. Then $a_1Ss \subseteq B_1$ or $a_1Ss' \subseteq B_1$. It implies that $(a_1Ss, a_2Ss) \subseteq B_1 \times A_2$ or $(a_1Ss', a_2Ss') \subseteq B_1 \times A_2$.

(2), (3) It follows from the method of the proof of (1).

(4) Let $(i, a)SsSs' \subseteq B_1 \sqcup B_2 = (1, B_1) \cup (2, B_2)$ for $s, s' \in S$, $(i, a) \in A_1 \sqcup A_2 = (1, A_1) \cup (2, A_2)$, and $i = 1, 2$. If $i = 1$ and $a \in A_1$, then $(1, a)SsSs' \subseteq B_1 \sqcup B_2$. Hence $aSsSs' \subseteq B_1$. Therefore $aSs \subseteq B_1$ or $aSs' \subseteq B_1$ since B_1 is a classical prime subact of A_1. Then $(1, a)Ss \subseteq B_1 \sqcup B_2$ or $(1, a)Ss' \subseteq B_1 \sqcup B_2$. If $i = 2$ and $a \in A_2$, then $(2, a)SsSs' \subseteq B_1 \sqcup B_2$. Similarly, we have $(2, a)Ss \subseteq B_1 \sqcup B_2$ or $(2, a)Ss' \subseteq B_1 \sqcup B_2$ since B_2 is a classical prime subact of A_2.

For the converse, let $a_1SsSs' \in B_1$ for $a_1 \in A_1$, $s, s' \in S$. Then $(1, a_1)SsSs' \subseteq B_1 \sqcup B_2$ and $(1, a_1) \in A_1 \sqcup A_2$. We conclude that $(1, a_1)Ss = (1, a_1Ss) \subseteq B_1 \sqcup B_2$ or $(1, a_1)Ss' = (1, a_1Ss') \subseteq B_1 \sqcup B_2$. Hence $a_1Ss \subseteq B_1$ or $a_1Ss' \subseteq B_1$ and B_1 is a classical prime subact of A_1. It follows from similar way for $B_2 \subseteq A_2$. □

The converse of third statement of pervious proposition does not hold in general. Notice this example:

Example 2.8. Let $(\mathbb{N}, .)$ be the natural numbers with respect to multiplication. The $\mathbb{N}$-subacts $2\mathbb{N}$ and $3\mathbb{N}$ of $\mathbb{N}$ are classical prime subacts of $\mathbb{N}$. But $2\mathbb{N} \times 3\mathbb{N}$ is not a classical prime subact of $\mathbb{N} \times \mathbb{N}$. This is because

$$(1, 1)\mathbb{N}2\mathbb{N}3 \subseteq 2\mathbb{N} \times 3\mathbb{N}$$

for $(1, 1) \in \mathbb{N} \times \mathbb{N}$ and $2, 3 \in \mathbb{N}$ while $(1, 1)\mathbb{N}2 \nsubseteq 2\mathbb{N} \times 3\mathbb{N}$ and $(1, 1)\mathbb{N}3 \nsubseteq 2\mathbb{N} \times 3\mathbb{N}$.

Definition 2.9. A homomorphism $f : A \to B$ is said to be *classical prime*, whenever $f(A)$ is a classical prime subact of B.

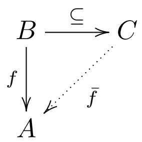
Now we study injectivity with respect to classical prime monomorphisms.

Definition 2.10. (1) An act A is said to be *classical prime injective* if for any classical prime monomorphism $g : B \to C$, any homomorphism $f : B \to A$ can be lifted to a homomorphism $\bar{f} : C \to A$, that is, $\bar{f}g = f$.
(2) An act A is said to be *weakly classical prime injective* if it is injective relative to embeddings of all classical prime right ideals into S.
(3) An act A is called *fg* (*cyclic*) *classical prime injective*, whenever for each classical prime homomorphism $g : F \to C$ from a finitely generated (cyclic) act F to an act C, and for any homomorphism $f : F \to A$, there exists a homomorphism $h : C \to A$ such that $hg = f$.

In [13], the concept of prime monomorphism is introduced similarly and also $\mathcal{M}$-injectivity of S-acts, where $\mathcal{M}$ is a subclass of all prime monomorphisms, is studied. Eventually, we are going to study $\mathcal{M}$-injectivity of S-acts, where $\mathcal{M}$ is a subclass of all classical prime monomorphisms.

Unquestionably, this definition is up to isomorphism and every object can be taken place by an isomorphic act. Therefore, we do not distinguish between classical prime monomorphism of acts and classical prime extensions; then we can assume that $g : B \to C$ is an inclusion and that B is a classical prime subact of C, up to isomorphism.

Lemma 2.11. An act A is (fg, cyclic) classical prime injective if and only if for any act C, for any (fg, cyclic) classical prime subact B, and for any homomorphism $f : B \to A$, there exists a homomorphism $\bar{f} : C \to A$ which extends f, that is, the diagram



commutes.

Equivalently, there are such the following results of fact on general $\mathcal{M}$-injectivity. Now, we can have following consequences where $\mathcal{M}$ is the class of all classical prime monomorphisms with the same proof in [11].

Proposition 2.12. An act A is weakly classical prime injective if and only if for any homomorphism $f : I \to A$, where $I \subseteq S$ is a classical prime right ideal of a monoid S, there exists an element $a \in A$ such that $f(s) = as$ for every $s \in I$.

In the next two lemmas, we give the relation between a classical prime injective act and its retract.

Lemma 2.13. Every retracts of (weakly) classical prime injective act is (weakly) classical prime injective.

Lemma 2.14. If an act A is classical prime injective and a classical prime subact of B, then A is a retract of B.

As a corollary, the foregoing lemma yields the desired conclusion.

Corollary 2.15. The following statements are equivalent for a monoid S:

(1) Every classical prime right ideal of S is a retract of S.

(2) Every classical prime right ideal of S is weakly classical prime injective.

Proof . (1) $\Rightarrow$ (2) Let P and I be classical prime right ideals of S and let $f : I \to P$ be a homomorphism. Then by assumption, there exists a homomorphism $g : S \to I$ such that $g_{|_I} = id_I$. Hence, f can be lifted to $fg : S \to P$.
(2) $\Rightarrow$ (1) It is clear by Lemma 2.14. □

Recall that every cofree act is injective and that every act can be embedded into an injective act; see Theorem 3.1.5 of [9]. Clearly, every injective act is classical prime injective. Therefore, every cofree act is classical prime injective and every act can be embedded into a classical prime injective act. It means that the category of **Act-S** has enough classical prime injective acts. By proving some considerable results, we are going to find that there exists a classical prime injective hull for any act.

Lemma 2.16. Every classical prime injective act contains a zero element.

Proof . Let A be a classical prime injective act. We claim that A is a classical prime subact of $A \cup \Theta$, where $\Theta = \{\theta\} \nsubseteq A$. Let $a \in A \cup \Theta$ and $s', s'' \in S$ be such that $aSs'Ss'' \subseteq A$. We are going to show that $aSs' \subseteq A$ or $aSs'' \subseteq A$. As $a \in A \cup \Theta$, we have $a \in A$ or $a = \theta$. If $a = \theta$, then $aSs'Ss'' = \Theta \nsubseteq A$. Also if $a \in A$, then $aSs' \subseteq A$ and $aSs'' \subseteq A$.

Now consider the natural embedding from A into $A \cup \Theta$, and the identity homomorphism id_A. Since A is classical prime injective, there exists a homomorphism $\bar{id}_A : A \cup \Theta \to A$, which extends id_A. Now $\bar{id}_A(\theta)s = \bar{id}_A(\theta s) = \bar{id}_A(\theta)$ for any $s \in S$. This means that $\bar{id}_A(\theta)$ is the zero element of A. □

Note that, in the Baer criterion for injectivity in the case of modules over a ring, it is sufficient to be hold injectivity relative to all inclusions of right ideals into the ring. Also the Skornjakov criterion, which says injectivity of an act is equivalent to being injective relative to all inclusions into cyclic acts; see [9]. Theorem 2.17 is the criterion for classical prime injectivity, which will be consequential in what follows.

Theorem 2.17. Assume that S is a commutative monoid and that any cyclic subact of an S-act is classical prime. An act with zero is classical prime injective if and only if it is injective relative to all inclusions of classical prime subact of cyclic acts.

Proof . Necessity. Obvious.
Sufficiency. Let A be an act with zero θ, which satisfies the assumption. Suppose that $g : F \to B$ is a classical prime monomorphism and that $f : F \to A$ is a homomorphism. We have to prove that there exists a homomorphism $\bar{f} : B \to A$ such that $\bar{f}g = f$. Consider
$\Sigma := \{(E, h) \mid F \subseteq E,\ E \textit{ is a classical prime subact of } B,$

$$h : E \to A \textit{ is a homomorphism extending } f\}.$$

Clearly $(F, f) \in \Sigma$; then it is nonempty. Define a partial order $\leq$ on Σ as follows:

$$(E_1, h_1) \leq (E_2, h_2) \Leftrightarrow E_1 \subseteq E_2 \textit{ and } h_{2_{|_{E_1}}} = h_1.$$

For any chain $\cdots \leq (E_i, h_i) \leq \cdots$ of Σ with $i \in I$, we claim that the pair $(\bigcup E_i, \bar{h})$, where $\bar{h}(e_i) = h_i(e_i)$ for any $e_i \in E_i$, is an upper bound. Then we show that $\bigcup E_i$ is a classical prime subact of B. Let $s_1, s_2 \in S$ and $b \in B$ be such that $bSs_1Ss_2 \subseteq \bigcup E_i$. Therefore $bs_1s_2 = b1s_11s_2 \in \bigcup E_i$. Thus there exists $j \in I$ such that $bs_1s_2 \in E_j$. Since S is a commutative monoid, we have $bSs_1Ss_2 \subseteq E_j$. Since E_j is a classical prime subact of B, we have $bSs_1 \subseteq E_j \subseteq \bigcup E_i$ or $bSs_2 \subseteq E_j \subseteq \bigcup E_i$. By Zorn's lemma, there exists a maximal element (C, k) in Σ. We claim that $C = B$.

Assume $C \neq B$. Then there exists $b \in B \setminus C$. Set $D = C \cup bS \supsetneq C$. Obviously $F \subseteq D = C \cup bS \subseteq B$. First suppose that $C \cap bS = \varnothing$, then set $D = C \sqcup bS$. We are going to show that D is a classical prime subact of B. Let $b'Ss'Ss'' \subseteq D$ for any $s', s'' \in S$ and $b' \in B$. Three cases may happen:

Case (1) If $b'Ss'Ss'' \subseteq bS$, then $b'Ss' \subseteq bS \subset D$ or $b'Ss'' \subseteq bS \subset D$, since bS is the classical prime subact of B by assumption.

Case (2) If $b'Ss'Ss'' \subseteq C$, then either $b'Ss' \subseteq C \subset D$ or $b'Ss'' \subseteq C \subset D$, since C is the classical prime subact of B.

Case (3) If the cases (1) or (2) do not happen, then there exist nonempty subsets $S_1 = \{t \in S \mid b'ts's'' \in C\}$ and $S_2 = \{t \in S \mid b'ts's'' \in bS\}$ of the commutative monoid S, such that $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \varnothing$, then $b'S_1s's'' \subseteq C$ and $b'S_2s's'' \subseteq bS$. Now whenever $1 \in S_1$, we have $b's's'' = b'1s's'' \in C$ and $b's's''S = b's's''SS = b'Ss'Ss'' \subseteq C$, since S is a commutative monoid. It implies that $S_2 = \varnothing$, which is a contradiction. If $1 \in S_2$, then with the same argument, we have $S_1 = \varnothing$, which is a contradiction again. Hence, case (3) does not occur. Therefore, D is a classical prime subact of B if $C \cap bS = \emptyset$.

Define a mapping $\bar{k} : D \to A$ such that $\bar{k}(c) = k(c)$ for any $c \in C$ and $\bar{k}(bs) = \theta$ for any $bs \in bS$. Then $\bar{k}$ is a homomorphism extending k and $(C, k) < (D, \bar{k})$ which contradicts the maximality of the pair (C, k).

Now suppose $C \cap bS \neq \varnothing$ and set $H = C \cap bS$. We are going to show that H is a classical prime subact of the cyclic act bS. Let $btSs'Ss'' \subseteq C \cap bS$ for $s', s'' \in S$ and $bt \in bS$. Hence, $btSs'Ss'' \subseteq C$ and $btSs'Ss'' \subseteq bS$. Since C is a classical prime subact of B, we can conclude that $btSs' \subseteq C$ or $btSs'' \subseteq C$. Hence, $btSs' \subseteq C \cap bS$ or $btSs'' \subseteq C \cap bS$. Set $\rho = k_{|_H}$. By hypothesis, A is classical prime injective relative to all inclusions of classical prime subacts to cyclic acts; then there exists a homomorphism $\bar{\rho} : bS \to A$ extending ρ. Define a mapping $\ell : D \to A$ such that $\ell(c) = k(c)$ for any $c \in C$ and $\ell(bs) = \bar{\rho}(bs)$ for any $bs \in bS$. If $x \in H$, then $\ell(x) = k(x)$ by the rule for $x \in C$ and also $\ell(x) = \bar{\rho}(x) = \rho(x) = k(x)$ by the rule for $x \in bS$. Hence, ℓ is well-defined. Obviously, ℓ is a homomorphism extending k. Again it contradicts the maximality of the pair (C, k). Hence $C = B$. □

If S is the one-element monoid $S = \{1\}$, then trivially every non-empty set A is a left and right S-act and any cyclic subact of A has just one element. Clearly, any cyclic subact of such an S-act is classical prime. We can guarantee the existence of such a monoid in the situation of Theorem 2.17.

Note that the category **Act-S** is complete and cocomplete and has all products, coproducts, pushouts, and pullbacks.

Proposition 2.18. Let $\{A_i : i \in I\}$ be a family of S-acts. Then

(1) $\prod_{i \in I} A_i$ is (fg) classical prime injective if and only if A_i is (fg) classical prime injective for all $i \in I$.

(2) If the coproduct $\coprod_{i \in I} A_i$ is (fg) classical prime injective, then each A_i is a (fg) classical prime injective act for all $i \in I$.

Proof . (1) By Lemma 2.16, every classical prime injective act has a zero element, then according to Proposition 2.1.4 of [9], each A_i is a retract of $\prod_{i \in I} A_i$. By Lemma 2.13, A_i is a classical prime injective act. Conversely, let $g : A \to B$ be a classical prime monomorphism and let $f : A \to \prod_{i \in I} A_i$ be a homomorphism; then we have homomorphisms $\rho_i f : A \to A_i$ for any $i \in I$, where ρ_i is the projection homomorphism. Since A_i is classical prime injective, there exists $h : B \to A_i$ for any $i \in I$ such that $hg = \rho_i f$. Hence, there exists a product induced homomorphism $\bar{h} : B \to \prod_{i \in I} A_i$ such that $\rho_i \bar{h} = h$. Therefore $\rho_i \bar{h} g = hg = \rho_i f$ and $\bar{h} g = f$.
(2) It is similar to (1). □

Recall that for a family $\{A_i : i \in I\}$ of S-acts with a unique fixed element θ, the direct sum $\bigoplus_{i\in I} A_i$ is defined as a subact of the product $\prod_{i\in I} A_i$ consisting of all $(a_i)_{i\in I}$ such that $a_i = \theta$ for all $i \in I$ except a finite number.

It is explicit that if $\bigoplus_{i\in I} A_i$ is classical prime injective, then by condition, each A_i contains zero. Therefore, A_i is a retract of $\bigoplus_{i\in I} A_i$. Hence, each A_i is classical prime injective, since the retract of each classical prime injective act is classical prime injective.

Proposition 2.19. Each direct sum of fg (cyclic) classical prime injective acts is fg (cyclic) classical prime injective.

Proof . Let $\{A_i\}_{i\in I}$ be a family of fg classical prime injective acts. Consider the following diagram:

$$\begin{array}{ccc} F & \xrightarrow{\;g\;} & C \\ {\scriptstyle f}\big\downarrow & & \\ \bigoplus_{i\in I} A_i & & \end{array}$$

where F is a finitely generated classical prime subact of C and let $\{x_1, x_2, \ldots, x_n\}$ be a generated set of F. Then $F = \bigcup_{j=1}^{n} x_j S$ and for a finite number J_j, we have $f(x_j) = (f(x_j)_i)_{i\in J_j}$ and for infinite number $i \in I \setminus J_j$, also we have $f(x_j) = (\theta_i)_{i\in I\setminus J_j}$. Consider $J = \bigcup_{j=1}^{n} J_j$ which is a finite set. Therefore $f(a)_i = \theta_i$ for every $a \in F$ and $i \in I \setminus J$. Hence $Im(f) \subseteq \bigoplus_{i\in J} A_i = \prod_{i\in J} A_i$. By Proposition 2.18 which holds for fg classical prime injective as well, each product of fg classical prime injective acts is a fg classical prime injective act; then there exists a homomorphism $h : C \to \bigoplus_{i\in J} A_i \hookrightarrow \bigoplus_{i\in I} A_i$, which extends f. $\square$

The converse of part (2) of Proposition 2.18, is more complicated. But we shall show in Proposition 2.21, its converse is true for a special monoid S.

Definition 2.20. A monoid S is called *left classical prime reversible* if $I \cap J \neq \varnothing$ for any classical prime right ideals I and J of S.

Let $(\mathbb{N}, .)$ be the natural numbers with respect to multiplication. All classical prime ideals of $\mathbb{N}$ are the form of $p\mathbb{N}$ where p is a prime number. For any prime numbers $p, q \in \mathbb{N}$, we have $pq \in p\mathbb{N} \cap q\mathbb{N}$. It means that we have $p\mathbb{N} \cap q\mathbb{N} \neq \varnothing$ for any prime numbers $p, q \in \mathbb{N}$ and $(\mathbb{N}, .)$ is a left classical prime reversible monoid.

Proposition 2.21. Assume that S is a commutative monoid and that any cyclic subact of any act is classical prime. The following statements are equivalent:

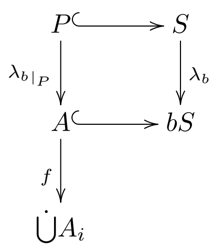
(1) All coproducts of classical prime injective acts are classical prime injective.

(2) $\{x, y\}$ is classical prime injective where x and y are fixed elements.

(3) S is a left classical prime reversible monoid.

Proof . (1) $\Rightarrow$ (2) It is obvious.
(2) $\Rightarrow$ (3) Suppose that S is not left classical prime reversible. Then there exist classical prime right ideals I and J of S such that $I \cap J = \varnothing$. We show that $I \cup J$ is a classical prime subact of S. Let $s_1Ss_2Ss_3 \subseteq I \cup J$ for any $s_1, s_2, s_3 \in S$. Then $s_1 1 s_2 1 s_3 = s_1s_2s_3 \in I \cup J$. Therefore $s_1s_2s_3 \in I$ or $s_1s_2s_3 \in J$. Consider $s_1s_2s_3 \in I$. Commutativity of the monoid S implies that $s_1Ss_2Ss_3 \subseteq I$ and then $s_1Ss_2 \subseteq I \subseteq I \cup J$ or $s_1Ss_3 \subseteq I \subseteq I \cup J$, since I is a classical prime right ideal. Let $f : I \cup J \to \{x, y\}$ by $f(i) = x, f(j) = y$ for any $i \in I$ and $j \in J$. We claim that f cannot be extended to S, and then $\{x, y\}$ is not a classical prime injective act. If there exists $\bar{f} : S \to \{x, y\}$ such that $\bar{f}_{|I \cup J} = f$, then $\bar{f}(1) = x$ or y, say $\bar{f}(1) = x$. Hence,$\bar{f}(t) = \bar{f}(1t) = \bar{f}(1)t = xt = x$ for any $t \in S$ and it contradicts $\bar{f}_{|I \cup J} = f$. Thus f cannot be extended to S.
(3) $\Rightarrow$ (1) Let S be a classical prime reversible monoid and let A_i be a classical prime injective act for any $i \in I$. By Lemma 2.16, any A_i contains a zero element, and then $\dot{\bigcup} A_i$ contains a zero element. Let A be a classical prime subact of the cyclic act bS and let $f : A \to \dot{\bigcup} A_i$ be a homomorphism.

Let $\lambda_b : S \to bS$ be an epimorphism defined by $\lambda_b(s) = bs$ for any $s \in S$ and let $P = \lambda_b^{-1}(A)$. Now we will show that P is a classical prime right ideal of S. Let $s_1, s_2, s_3 \in S$ and let $s_1Ss_2Ss_3 \subseteq \lambda_b^{-1}(A)$; then $\lambda_b(s_1Ss_2Ss_3) \subseteq A$. Therefore $bs_1Ss_2Ss_3 \subseteq A$. As A is a classical prime subact of bS and $bs_1 \in bS$, we have $bs_1Ss_2 \subseteq A$ or $bs_1Ss_3 \subseteq A$, which means that, $\lambda_b^{-1}(bs_1Ss_2) = \{\lambda_b^{-1}(bs_1ts_2) \mid \forall t \in S\} = \{\ s_1ts_2 \mid \forall t \in S\} = s_1Ss_2 \subseteq \lambda_b^{-1}(A) = P$ or $\lambda_b^{-1}(bs_1Ss_3) = s_1Ss_3 \subseteq \lambda_b^{-1}(A) = P$. Consider this diagram



Suppose that there exist $i, j \in I$, $i \neq j$, with $f(A) \cap A_i \neq \varnothing$ and $f(A) \cap A_j \neq \varnothing$. Hence, $f(A)$ is decomposable. It is obvious to show that A and then $P = \lambda_b^{-1}(A)$ are decomposable. Hence, there exist right ideals J and K such that $P = J \cup K$ and $J \cap K = \varnothing$. Now we claim that right ideals J and K are classical prime right ideals of S and it contradicts the left classical prime reversibility of S. We show that J is a classical prime right ideal. Let $s_1, s_2, s_3 \in S$ be such that $s_1Ss_2Ss_3 \subseteq J \subseteq P$. Then $s_1Ss_2 \subseteq P$ or $s_1Ss_3 \subseteq P = J \cup K$. The following two cases may happen:

Case (1) If $s_1Ss_2 \subseteq J$ or $s_1Ss_3 \subseteq J$, we do not have any things to prove.

Case (2) If $s_1Ss_2 \nsubseteq J$ and $s_1Ss_3 \nsubseteq J$, then $s_1Ss_2 \cap K \neq \varnothing$ and $s_1Ss_3 \cap K \neq \varnothing$, then there exists an element $t \in S$ such that $s_1ts_2 \in K$. Since K is a right ideal of S, then we can conclude that $s_1ts_2rs_3 \in K$ for any $r \in S$. Hence, $s_1ts_2rs_3 \in J \cap K$, which contradicts the nullity of the intersection of the two right ideals J and K.

The argument above implies the existence of $i \in I$ such that $f(A) \subseteq A_i$. Since A_i is classical prime injective by assumption, then f can be extended to a homomorphism $\bar{f} : bS \to \dot{\bigcup} A_i$. Thus $\dot{\bigcup} A_i$ is classical prime injective by Theorem 2.17. $\square$

Recall that for a category $\mathcal{A}$ and a subclass $\mathcal{M}$ of monomorphisms in $\mathcal{A}$, we say that an object A satisfies the $\mathcal{M}$-transferability property if any diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle g}\downarrow & & \\ B & & \end{array}$$

with $f \in \mathcal{M}$ can be completed to a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} \\ B & \xrightarrow[h]{} & D \end{array}$$

with $h \in \mathcal{M}$; see [3].

Remark 2.22. Note that if pushouts exist in a category $\mathcal{A}$ and $\mathcal{M}$ is left cancellable, then it is easily seen that pushouts transfer (or preserve) $\mathcal{M}$-morphisms if and only if $\mathcal{A}$ has the $\mathcal{M}$-transferability property.

Now, we are going to show that the subclass of all classical prime monomorphisms is left cancellable.

Lemma 2.23. If fg is classical prime monomorphism, then g is the same.

Proof . Let $g : A \to B$ and $f : B \to C$ be homomorphisms for acts A, B, and C such that fg is a classical prime monomorphism. We show that g is a classical prime monomorphism. Let $b \in B$ and $s, s' \in S$ be such that $bSsSs' \subseteq g(A)$. Therefore $f(b)SsSs' = f(bSsSs') \subseteq fg(A)$. Since $fg(A)$ is a classical prime subact of C and $f(b) \in C$, we have $f(b)Ss \subseteq fg(A)$ or $f(b)Ss' \subseteq fg(A)$. Suppose $f(b)Ss \subseteq fg(A)$. Thus for any $t \in S$, there exists $a \in A$ such that $f(b)ts = f(bts) = fg(a)$. Then for any $t \in S$, there exists $a \in A$ such that $bts = g(a)$, which implies that $bSs \subseteq g(A)$. If $f(b)Ss' \subseteq fg(A)$ then also $bSs' \subseteq g(A)$. Hence $g(A)$ is a classical prime subact of B. □

In **Act-S**, $\mathcal{M}$-transferability property, where $\mathcal{M}$ is a class of classical prime monomorphisms, is equivalent to that pushout transfers a classical prime monomorphism, since every classical prime monomorphism is left cancellable.

Theorem 2.24. Pushouts transfer classical prime monomorphisms.

Proof . We have pushouts transfer monomorphisms by [12, Theorem 1]. For transferring the classical prime monomorphisms, take the classical prime monomorphism $f : A \to B$ and the homomorphism $g : A \to C$ and $D = \frac{B\dot{\cup}C}{\rho}$ where ρ is the congruence relation on $B\dot{\cup}C$ generated by all pairs $(g(a), f(a))$ for all $a \in A$ and $B\dot{\cup}C = (\{1\} \times B) \cup (\{2\} \times C)$. Further, take $q_C = \pi u_C$ and $q_B = \pi u_B$, where π is the canonical epimorphism onto $B\dot{\cup}C$ and u_C and u_B are injections. Then $((q_C, q_B), D)$ is the pushout of the pair (g, f) in **Act-S**.

Since f is a classical prime monomorphism, we have that $f(A) \subseteq B$ is a classical prime subact. We are going to show that q_C is a classical prime monomorphism, or equivalently $q_C(C) = \{q_C(c)|c \in C\} = \{\pi u_C(c)|c \in C\} = \{(2, [c]_\rho)|c \in C\}$ is a classical prime subact of D. For any $s_1, s_2 \in S$ and $[(i, d)]_\rho \in D$, consider $[(i, d)]_\rho Ss_1Ss_2 \subseteq q_C(C)$ such that if $i = 1$ or 2, then $d \in B$ or $d \in C$, respectively. We are going to show that $[(i, d)]_\rho Ss_1 \subseteq q_C(C)$ or $[(i, d)]_\rho Ss_2 \subseteq q_C(C)$.

If $[(i, d)]_\rho \in q_C(C)$, then we do not have anything to prove. Therefore, suppose $[(i, d)]_\rho \notin q_C(C)$. It implies that there exists an element $b' \in B$ such that $[(i, d)]_\rho = [(1, b')]_\rho$. We have $[(1, b')]_\rho Ss_1Ss_2 \subseteq \{[(1, b)]_\rho \mid b \in B\}$ and on the other hand, $[(1, b')]_\rho Ss_1Ss_2 \subseteq q_C(C)$. Thus

$$[(1, b')]_\rho Ss_1Ss_2 \subseteq q_C(C) \cap \{[(1, b)]_\rho \mid b \in B\} = \{[(1, b)]_\rho \mid b \in f(A)\}.$$

Therefore $b'Ss_1Ss_2 \subseteq f(A)$. Since $f(A)$ is a classical prime subact of B, then $b'Ss_1 \subseteq f(A)$ or $b'Ss_2 \subseteq f(A)$. Hence,

$$(1, [b']_\rho Ss_1) \subseteq \{(1, [f(a)]_\rho) \mid a \in A\ \} = \{\ (2, [g(a)]_\rho) \mid a \in A\ \} \subseteq q_C(C)$$

or

$$(1, [b']_\rho Ss_2) \subseteq \{(1, [f(a)]_\rho) \mid a \in A\ \} = \{\ (2, [g(a)]_\rho) \mid a \in A\ \} \subseteq q_C(C).$$

It means that $q_C(C)$ is a classical prime subact of $\frac{B\dot{\cup}C}{\rho}$. □

Definition 2.25. An act A is called *classical prime retract* of an act B if for every classical prime monomorphism $h : A \to B$, there exists a homomorphism $g : B \to A$ such that $gh = id_A$.

Equivalently, A is called *classical prime retract* of B if A be a classical prime subact of B and there exists a homomorphism $g : B \to A$ such that $g_{|_A} = id_A$.

An S-act A is called *absolute classical prime* retract if it is a retract of each classical prime extensions.

A subact B of A is called essential in A if and only if any homomorphism $f : A \to H$, where H is any S-act with restriction to B is monomorphism , then f is itself monomorphism [5].

We call a classical prime extension A of B with the embedding $f : B \to A$ a *essential classical prime extension* of B if Imf is essential in A.

A classical prime extension A of an act B is called *classical prime injective hull* of B if it is an essential classical prime extension of B, which is also classical prime injective.

The next theorem is one of the most interesting theorems about injectivity of acts with respect to any subclass of monomorphisms. This was proved by Berthiaume in [5], for injective acts and Banaschewski in [2] and Barzegar in [3] proved it for $\mathcal{M}$-injective acts when $\mathcal{M}$ is a subclass of monomorphisms. Here is what we have for the class of classical prime monomorphisms.

Theorem 2.26. The following are equivalent for an S-act A:

(1) A is classical prime injective.

(2) A is an absolute classical prime retract.

(3) A has no proper essential classical prime extension.

Now according to [2] and Theorem 2.24, the next corollary immediately results from the above theorem.

Corollary 2.27. Every act has a classical prime injective hull.

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