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# Reverse minisum single facility location problem with variable weights

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**Abstract**

In the classical minisum facility location problem, the goal is to find the placement of a new facility that minimizes the sum of weighted distances to a given set of client points. In contrast, the reverse minisum single facility location problem assumes a fixed facility location and focuses on adjusting the weights of the client points. The objective is to improve the weighted distances between the facility and clients, subject to a budget constraint on weight modifications. This paper introduces an $O(n \log n)$ algorithm for the reverse problem with variable weights, applicable to both network and continuous location models. Experimental results on diverse instances demonstrate the algorithm's effectiveness.

**Keywords:** Reverse facility location, Variable weights, Continuous facility location, Minisum.
**MSC :** Primary: 90B80; Secondary: 90B06, 90C35.

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## 1 Introduction

Reverse and inverse single facility location problems represent a fascinating and increasingly relevant area of research in operations research, logistics, and spatial analysis. These problems extend the classical facility location framework by addressing scenarios where the goal is to infer unknown parameters or optimize facility locations based on observed data or specific objectives. While traditional facility location problems focus on determining the optimal placement of facilities to serve a given set of demand points, reverse and inverse problems take a different approach, often involving the estimation of demand points, weights, or other parameters based on observed patterns or desired outcomes.

Typically, the facility location models are categorized into three types: continuous, discrete, and network location models. In continuous models, the goal is to plan facility locations within a two-dimensional space $R^2$. Discrete location models, on the other hand, restrict facility placement to specific predetermined points. Network location models require facilities to be situated on a defined network. Specifically, the continuous version of the single facility location problem seeks to identify a facility's position in a plane that minimizes client servicing costs. .

The minisum facility location problem, also known as the Weber problem, has a rich history dating back to the early 20th century. The classical formulation seeks to find the optimal location of a facility that minimizes the total weighted distance to a set of demand points, where the weights are fixed and represent the demand or importance of each location. This problem has been extensively studied, with numerous extensions and variations proposed to address different real-world scenarios (see e.g. [16]). The network based version of minisum facility location problem is called median problem.

In some real-world applications, facilities may already exist, and the challenge lies in optimizing their current locations by adjusting certain parameters. When the goal is to modify these parameters at minimal cost to make the given locations optimal, the problem is referred to as an inverse location problem. Conversely, if the objective is to improve the given locations as much as possible within a specified budget constraint, the problem is termed a reverse location problem.

The inverse minisum problem has been extensively studied by various researchers. For instance, Burkard et al. [10] proposed an $O(n \log n)$ algorithm for the inverse median problem on trees. Galavii [14] later improved the time complexity of this problem to linear time. Burkard et al. [11] developed an $O(n^2)$ algorithm for the inverse median problem on cycles. Guan and Zhang [15] explored the inverse median problem on trees using Chebyshev and Hamming norms. Sepasian and Rahbarnia [22] later solved the inverse median problem with varying vertex weights and edge lengths on trees. Nguyen [20] presented an algorithm for the inverse median problem with variable vertex weights on block graphs. This problem with variable edge lengths and weights has also been investigated by Baroughi et al. [4]. Their research indicates that the problem's complexity escalates when variable parameters are introduced, requiring more advanced algorithms to attain optimal solutions. In a similar vein, Sepasian and Rahbarnia [22] explored variable vertex weights and edge reductions, further demonstrating the flexibility of the inverse median problem in adapting to diverse network configurations. Alizadeh et al. [1] proposed combinatorial algorithms for inverse obnoxious median problem.

The continuous variant of the inverse facility location problem has garnered less research focus compared to its graph-based counterpart. Some studies have addressed the inverse minisum single facility location problem with variable weights, as evidenced in works [10, 7]. Burkard et al. [10] developed an algorithm that achieves a time complexity of $O(n \log n)$ for this problem under the rectilinear norm for distance measurement. On the other hand, when employing the Euclidean norm, Burkard et al. [7] formulated a linear programming model and showed that the unit-cost model can also be resolved in $O(n \log n)$ time. Furthermore, Baroughi-Bonab et al. [3] investigated the inverse minisum single facility location problem with variable coordinates, proving that problems involving rectilinear and Chebyshev norms are NP-hard. A general method for solving inverse location problems in the plane have been proposed by Fathali [12]. He showed that his presented method efficiently solve the inverse minisum single facility location problem. His method is applied to solve the problem with variable coordinates by Tour-Savadkoohi and Fathali [23].

On the other hand, the reverse median problem on general networks with variable edge lengths are known to be NP-hard [5, 8, 24]. Berman et al. [5] studied the reverse median problem with variable edge lengths on trees, while Burkard et al. [8] provided a linear time algorithm for this problem on cycles. Zhang et al. [24] considered reverse median problem with variable edge lengths and proposed a linear programming model for this problem on trees. Some other reverse location problems also have been investigated by authors. For instance, Nazari and Fathali [17] presented a meta-heuristic algorithm to solve the reverse backup 2-median problem with variable coordinates. Nguyen [19] considered the reverse 1-center problem. Omidi et al. [21] and Nazari and Fathali [18] proposed polynomial time algorithms for the reverse balanced 2-facility location problems with variable edge lengths and variable points weights, respectively.

Despite these developments, to the best of our knowledge, the reverse minisum problem with variable weights has not yet been explored by any researcher. To address this gap, this paper focuses on the reverse minisum single facility location problem with variable weights.

This paper is organized as follow: A description of minisum single facility location problem and its inverse and reverse models are provided in Section 2. Section 3 contains the propose algorithm for solving this problem. The computational results are given in Section

4.

## 2 Problem definition

Let $A$ be the set contains $n$ given disjoint points $\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_n$ which are the locations of clients and for $i = 1, ..., n$, $w_i \geq 0$ be the weight of point $\mathbf{P}_i$. In a minisum single facility location problem, the goal is finding a point $\mathbf{x}$ such that the following objective function is minimized,

$$f(\mathbf{x}, \mathbf{w}) = \sum_{\mathbf{P}_i \in A} w_i d(\mathbf{P}_i, \mathbf{x}), \tag{2.1}$$

where $d(\mathbf{P}_i, \mathbf{x})$ is the distance between points $\mathbf{P}_i$ and $\mathbf{x}$.

In continuous location models, points are located in the plane, and their distances are measured using specific norms. For example, if distances are measured by an $L_p$ norm, the distance between points $\mathbf{P}_i = (a_i, b_i)$ and $\mathbf{x} = (x, y)$ is calculated as follows:

$$d(\mathbf{P}_i, \mathbf{x}) = (|x - a_i|^p + |y - b_i|^p)^{\frac{1}{p}}.$$

In this case, the following property is proven, see e.g. [16].

**Lemma 2.1.** The optimal solution of the minisum single facility location problem in the plane lies in the convex hull of existing points.

The model (2.1) with $L_2$ norm can be solved by an iterative method, called Weiszfeld's algorithm. This algorithm iteratively updates the estimate of $\mathbf{x}$ using a weighted average of the demand points, where weights are inversely proportional to the current distance. It starts with an initial guess $\mathbf{x_0}$ and find the new point using the following relation:

$$x_{k+1} = \frac{\sum_{i=1}^{n} \frac{w_i a_i}{d(\mathbf{P}_i, \mathbf{x_k})}}{\sum_{i=1}^{n} \frac{w_i}{d(\mathbf{P}_i, \mathbf{x_k})}}, \quad y_{k+1} = \frac{\sum_{i=1}^{n} \frac{w_i b_i}{d(\mathbf{P}_i, \mathbf{x_k})}}{\sum_{i=1}^{n} \frac{w_i}{d(\mathbf{P}_i, \mathbf{x_k})}}.$$

On the other hand, in the network version of location problems, points are situated on the vertices of the network, and $d(\mathbf{P}_i, \mathbf{x})$ represents the length of the shortest path between $\mathbf{P}_i$ and $\mathbf{x}$.

Now consider the case of minisum single facility location problem, in which a predetermined point $\bar{\mathbf{x}}$ is provided and the goal is to adjust the weights of the given points $\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_n$ such that the following objective functions are minimized.

$$g_1(\mathbf{r}, \mathbf{s}) = \sum_{i=1}^{n} (c_i^+ r_i + c_i^- s_i),$$

$$g_2(\mathbf{r}, \mathbf{s}, \bar{\mathbf{x}}) = \sum_{i=1}^{n} (w_i + r_i - s_i) d(\mathbf{P}_i, \bar{\mathbf{x}}),$$

where for $i = 1, ..., n$, $r_i$ and $s_i$ represent the amounts of increase and decrease of $w_i$, respectively, with $\mathbf{r} = (r_1, ..., r_n)$, $\mathbf{s} = (s_1, ..., s_n)$. Also for $i = 1, ..., n$, $c_i^+ \geq 0$ and $c_i^- \geq 0$ denote the unit costs of increasing and decreasing $w_i$, respectively.

Two approaches are considered to solve this bi-objective problem:

1. Minimizing $g_1(\mathbf{r}, \mathbf{s})$ such that $g_2(\mathbf{r}, \mathbf{s}, \bar{\mathbf{x}})$ is less than or equal to $g_2(\mathbf{r}, \mathbf{s}, \mathbf{x})$ for any $\mathbf{x}$.
2. Minimizing $g_2(\mathbf{r}, \mathbf{s}, \bar{\mathbf{x}})$ such that $g_1(\mathbf{r}, \mathbf{s})$ is less than a given budget $B$.

These approaches are called inverse and reverse minisum single facility location problems in literature, respectively. Note that the goal in the inverse problem is adjusting the weights of points with minimum cost such that $\bar{\mathbf{x}}$ becomes an optimal solution of problem (2.1). On the other hand, in the reverse problem the aim is improving $f(\bar{\mathbf{x}}, \mathbf{w})$ as much as possible such that the total cost of modifying the weights of points does not exceed a given budget.

Inverse location problems are more commonly applied in practice, particularly in logistics, urban planning, and supply chain management, where decision-makers need to justify or slightly adjust existing solutions rather than redesigning entire networks. On the other hand, reverse location problems are often appearing in competitive scenarios (e.g., retail competition) or robustness analysis (e.g., identifying vulnerabilities in emergency service coverage).

Considering Lemma 2.1, the optimal solution of continuous minisum single facility location problem is in the convex hull of existing points. Thus, for the inverse model we conclude the following lemma.

**Lemma 2.2.** If the given point $\bar{\mathbf{x}}$ does not lie in the convex hull of $\mathbf{P}_1, ..., \mathbf{P}_n$, then the inverse model with variable weights is infeasible.

However, since in the reverse location problem, the goal is not to transform a given point $\bar{\mathbf{x}}$ into an optimal solution, but rather to improve the objective value of the minisum problem as much as possible, respect to the given budget constraints. Thus, the Lemma 2.2 does not apply to reverse minisum location problem with variable weights, and the following property can be stated.

**Lemma 2.3.** If $B \geq \min_{i=1,\ldots,n}\{c_i^+, c_i^-\}$. Then the reverse minisum single facility location problem with variable weights in the plane is feasible for any given point $\bar{\mathbf{x}}$.

In this paper, we consider the reverse model, that is:

$$\mathbf{P}: \qquad \min \ g_2(\mathbf{r},\mathbf{s},\bar{\mathbf{x}}) = \sum_{i=1}^{n}(w_i + r_i - s_i)d(\mathbf{P}_i,\bar{\mathbf{x}}), \tag{2.2}$$

$$s.t.$$

$$\sum_{i=1}^{n}(c_i^+ r_i + c_i^- s_i) \leq B, \tag{2.3}$$

$$0 \leq r_i \leq u_i, \quad i = 1, 2, ..., n, \tag{2.4}$$

$$0 \leq s_i \leq w_i, \quad i = 1, 2, ..., n, \tag{2.5}$$

where $u_i$, for $i = 1, ...n$, is an upper bound for the value of increasing $w_i$. Since the value of $d(\mathbf{P}_i, \bar{\mathbf{x}})$ is known for both continuous and network cases, the model $\mathbf{P}$ can be applied to the reverse problem in both scenarios.

## 3 Solving approach

In this section, we show the model $\mathbf{P}$ can be converted to a continuous knapsack problem, which can be solved in a polynomial time.

First consider that objective function of model $\mathbf{P}$ can be written as follows:

$$g_2(\mathbf{r},\mathbf{s},\bar{\mathbf{x}}) = \sum_{i=1}^{n} w_i d(\mathbf{P}_i,\bar{\mathbf{x}}) + \sum_{i=1}^{n}(r_i - s_i)d(\mathbf{P}_i,\bar{\mathbf{x}}).$$

Since, for $i = 1, ..., n$, $w_i$, $\mathbf{P}_i$, and $\bar{\mathbf{x}}$ are given, then the first term is constant and can be removed.

**Lemma 3.1.** Let $(\mathbf{r}^*, \mathbf{s}^*)$ be an optimal solution of model $\mathbf{P}$, then $\mathbf{r}^* = 0$.

**Proof .** Let $(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ be a feasible solution of model $\mathbf{P}$, such that $\bar{\mathbf{r}} \neq 0$. Thus $\sum_{i=1}^{n}(c_i^+ \bar{r}_i + c_i^- \bar{s}_i) \leq B$. Consider the new solution $(0, \bar{\mathbf{s}})$. Since $c_i^+ \geq 0$, this solution is also feasible for model $\mathbf{P}$. On the other hand, since we supposed that the the given points $\mathbf{P}_i$ are disjoint, then for $i = 1, ..., 0$, $d(\mathbf{P}_i, \bar{\mathbf{x}}) \geq 0$, and at least one of these distances is positive, thus

$$\sum_{i=1}^{n} -\bar{s}_i d(\mathbf{P}_i,\bar{\mathbf{x}}) < \sum_{i=1}^{n}(\bar{r}_i - \bar{s}_i)d(\mathbf{P}_i,\bar{\mathbf{x}}).$$

Therefore, $g_2(0, \bar{\mathbf{s}}, \bar{\mathbf{x}}) < g_2(\bar{\mathbf{r}}, \mathbf{s}, \mathbf{x})$, which means $(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ is not optimal. □

**Corollary 3.2.** To solve the reverse minisum single facility location problem with variable weights, it suffices to decrease the weight of the points.

Considering above discussion, the model $\mathbf{P}$ can be reduced to the following model.

$$\mathbf{P_r}: \qquad \max \;\; g_3(\mathbf{s}, \bar{\mathbf{x}}) = \sum_{i=1}^{n} s_i d(\mathbf{P}_i, \bar{\mathbf{x}}), \tag{3.1}$$

$$s.t.$$

$$\sum_{i=1}^{n} c_i^- s_i \leq B, \tag{3.2}$$

$$0 \leq s_i \leq w_i, \quad i = 1, 2, ..., n \qquad . \tag{3.3}$$

Model $\mathbf{P}_r$ is a continuous knapsack model. Note that although the 0/1 knapsack problem, i.e. the problem with $s_i \in \{0, 1\}$, is NP-hard, see [2], the continuous version of knapsack problem can be solved in $O(n \log n)$ time. The 0/1 case has been solved using various methods, ranging from exact algorithms such as branch and bound (for small instances) to heuristics and approximation techniques (for large-scale problems). The continuous knapsack problem is efficiently solved using a greedy algorithm (sort by ratio of objective coefficient to constraint coefficient). It guarantees an optimal solution in $O(n \log n)$ time. Algorithm $A_1$ shows the steps of solving model $\mathbf{P}_r$ using knapsack method.

**Algorithm** $[A_1]$.
**Input:** The point $\bar{\mathbf{x}}$ and existing points $\mathbf{P}_i$ and their weights $w_i$, for $i = 1, ..., n$. Also, $c_i^-$ the cost of decreasing of vertices weights and the budget $B$ are given.
**Output:** The values of decreasing vertices wights $s_i$.

1. **For** $i = 1, ..., n$ do
   (a) **Calculate** $d(\mathbf{P}_i, \bar{\mathbf{x}})$, the distance between $\mathbf{P}_i$ and $\bar{\mathbf{x}}$.
   (b) **Set** $dc_i = \frac{d(\mathbf{P}_i, \bar{\mathbf{x}})}{c_i^-}$.
   **end for**
2. **Sort** the points in descending order based on their corresponding $dc_i$ and call their new indices by $i_1, ..., i_n$.
3. **Set** $j = 1$.
4. **While** $B > 0$ **do** the following:
   (a) **Set** $s_{i_j} := min\{\frac{B}{c_{i_j}^-}, w_{i_j}\}$.
   (b) **Set** $B := B - s_{i_j} c_{i_j}^-$.
   (c) **Set** $j := j + 1$.
   **end while**
5. **Set** $g_4(\mathbf{s}, \bar{\mathbf{x}})$=$f(\bar{\mathbf{x}}, \mathbf{w}) - g_3(\mathbf{s}, \bar{\mathbf{x}})$.

**End of algorithm**

In this algorithm, $g_4(s, \bar{\mathbf{x}})$ indicates the difference between the initial and final values of objective function of minisum single facility location problem.

Note that sorting step can be done in $O(n \log n)$ time by merge sort method and step 4, needs at most $O(n)$ time. Thus the time complexity of the Algorithm $A_1$ is $O(n \log n)$ and we conclude the following theorem.

**Theorem 3.3.** The reverse minisum single facility location problem with variable vertices weights can be solved in $O(n \log n)$ time.

## 4 Numerical examples

In this section we examine some test problems on our presented algorithm. All the experiments were run on a PC with Intel Core i3 processor, 4 GB of RAM and CPU 2.7 GHz. The proposed algorithm for reverse minisum single facility location problem was tested on 3 test problems with varying given points and budgets.

For the first test problem, refer to Table 1, which provides the data for an instance of the reverse minisum location problem. The coordinates and weights of these points are sourced from [6]. Also, all costs and upper bounds are randomly selected in the range $[1, 10]$. Let the given point be $\bar{\mathbf{x}} = (\bar{x}, \bar{y})$. We examine Algorithm $A_1$ to find the solution of this instance for $L_2$ norm. The results for different $\bar{\mathbf{x}}$ are shown in Table 2. In this table, the row labeled $\mathbf{x}_k$ obtained by the Weiszfeld's Algorithm [16] refers to the solution of problem 2.1 with respect to $\hat{w}_i = w_i + r_i - s_i$ for $i = 1, ..., n$.

| $(a_i, b_i, w_i, c_i^+, c_i^-, u_i)$ | $(a_i, b_i, w_i, c_i^+, c_i^-, u_i)$ | $(a_i, b_i, w_i, c_i^+, c_i^-, u_i)$ |
|---|---|---|
| $(1, 2, 3, 1.5, 2, 1)$ | $(4, 4, 1, 2, 1.7, 4)$ | $(7, 1, 2, 4, 2, 2)$ |
| $(1, 3, 2, 3, 1, 3)$ | $(4, 9, 2, 2.5, 3, 1)$ | $(7, 2, 3, 3, 1, 4)$ |
| $(2, 5, 1, 6, 4, 5)$ | $(5, 3, 2, 1, 5, 2)$ | $(8, 5, 1, 2, 3, 5)$ |
| $(3, 6, 3, 3, 1, 7)$ | $(5, 5, 1, 2.5, 1, 4)$ | $(8, 8, 3, 3, 4, 3)$ |
| $(4, 8, 2, 1, 2, 3)$ | $(6, 6, 3, 3, 2, 5)$ | $(9, 7, 3, 1, 2.5, 8)$ |
| $(4, 1, 3, 5, 4, 5)$ | $(6, 3, 3, 4, 3, 4)$ | $(9, 6, 2, 1.5, 4, 2)$ |

Table 1: Data for the problem with 18 points.

We also assessed the proposed algorithm using three additional instances: Ruspini 75 , Bongartz 287 and TSPLIB 654, which can be found in Beasley [6]. The coordinate ranges for the points in these instances are provided in the third column of Table 3. All the weights and costs are randomly generated in the interval $[1, 10]$.

The results of testing our algorithm on the instances in the case of Euclidean norm, are shown in Table 4.

Table 2: The results of Algorithm $A_1$ for the instance with 18 points and Euclidean norm.

| $\bar{\mathbf{x}}$ | $(2,2)$ | $(-3,-5)$ | $(7,7)$ |
|---|---|---|---|
| $\mathbf{x}_k$ | $(3.9827, 2.6475)$ | $(5.5629, 3.4575)$ | $(6.0714, 5.6847)$ |
| B | 54 | 21 | 17 |
| $\hat{w}_1$ | 3 | 3 | 0 |
| $\hat{w}_2$ | 2 | 0 | 0 |
| $\hat{w}_3$ | 1 | 1 | 1 |
| $\hat{w}_4$ | 0 | 0 | 0 |
| $\hat{w}_5$ | 0 | 0 | 2 |
| $\hat{w}_6$ | 3 | 3 | 3 |
| $\hat{w}_7$ | 1 | 1 | 1 |
| $\hat{w}_8$ | 0 | 2 | 2 |
| $\hat{w}_9$ | 2 | 2 | 2 |
| $\hat{w}_{10}$ | 0 | 0 | 1 |
| $\hat{w}_{11}$ | 0 | 0 | 3 |
| $\hat{w}_{12}$ | 3 | 3 | 3 |
| $\hat{w}_{13}$ | 0 | 2 | 0.5 |
| $\hat{w}_{14}$ | 0 | 0 | 0 |
| $\hat{w}_{15}$ | 0 | 1 | 1 |
| $\hat{w}_{16}$ | 0 | 3 | 3 |
| $\hat{w}_{17}$ | 0 | 2.2 | 3 |
| $\hat{w}_{18}$ | 0.875 | 2 | 2 |
| $f(\bar{\mathbf{x}}, \mathbf{w})$ | 1.9714e+02 | $5.1430e + 02$ | $1.6584e + 02$ |
| $f(\mathbf{x}_k, \hat{\mathbf{w}})$ | 38.112 | 86.963 | 81.273 |
| $g_4(\mathbf{r}, \mathbf{s}, \bar{\mathbf{x}})$ | 44.113 | 323.66 | 91.624 |
| $CPU(in\ \ sec)$ | 0.0125 | 0.0125 | 0.0128 |

Table 3: The coordinate ranges of existing points of instances from [6].

| Instance | $n$ | Coordinate ranges |
|---|---|---|
| Ruspini | 75 | $(4,4)$ *to* $(117,156)$ |
| Bongartz | 287 | $(5,5)$ to $(48,48)$ |
| TSPLIB | 654 | $(1000,1000)$ to $(5000,5000)$ |

Table 4: The results of Algorithm $A_1$ for the instances from [6] with Euclidean norm.

| Instance | $n$ | $\bar{\mathbf{x}} = (\bar{x}, \bar{y})$ | $B$ | $\mathbf{x}_k = (x_k, y_k)$ | $g_4(\mathbf{r}, \mathbf{s}, \bar{\mathbf{x}})$ | $f(\bar{\mathbf{x}}, \mathbf{w})$ | $f(\mathbf{x}_k, \hat{\mathbf{w}})$ | $CPU$ |
|---|---|---|---|---|---|---|---|---|
| Ruspini | 75 | $(50, 50)$ | 112 | $(34.748, 74.829)$ | $1.0049e + 04$ | $1.4562e + 04$ | $8.6339e + 03$ | 0.0204 |
| | | $(-80, -20)$ | 235 | $(27.706, 72.131)$ | $2.1346e + 04$ | $3.9671e + 04$ | $6.2939e + 03$ | 0.0109 |
| | | $(-20, 80)$ | 190 | $(28.807, 75.437)$ | $1.0760e + 04$ | $1.8866e + 04$ | $6.9305e + 03$ | 0.0102 |
| Bongartz | 287 | $(15, 35)$ | 400 | $(21.666, 30.87)$ | $5.3305e + 03$ | $7.7422e + 03$ | $3.6718e + 03$ | 0.0128 |
| | | $(20, 29)$ | 100 | $(22.341, 30.821)$ | $4.9331e + 03$ | $5.7307e + 03$ | $4.5411e + 03$ | 0.0117 |
| | | $(50, 45)$ | 1200 | $(23.947, 30.136)$ | $1.0231e + 04$ | $2.3024e + 04$ | $2.3089e + 03$ | 0.0139 |
| TSPLIB | 654 | $(2000, 4000)$ | 2000 | $(2118.4, 3757.3)$ | $2.4774e + 06$ | $5.1197e + 06$ | $2.4645e + 06$ | 0.0121 |
| | | $(1500, 1500)$ | 5000 | $(1125.3, 2686.9)$ | $1.2712e + 06$ | $6.9986e + 06$ | $8.5785e + 05$ | 0.0140 |
| | | $(3500, 3500)$ | 6000 | $(4138.7, 3665.6)$ | $7.4573e + 05$ | $4.8467e + 06$ | $7.2379e + 05$ | 0.0117 |

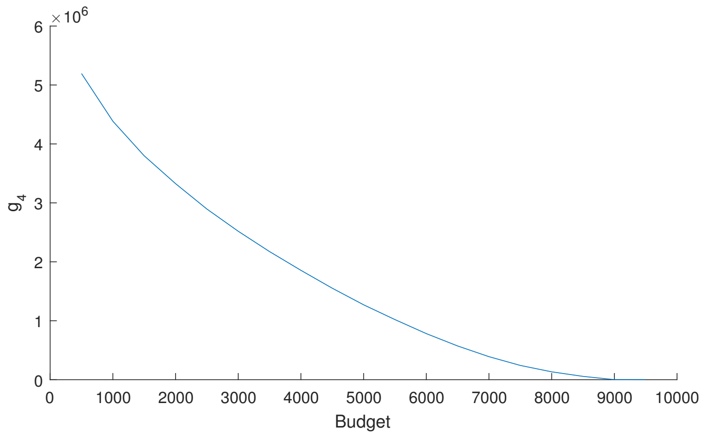


Figure 1: The results of Algoritm $A_1$ for TSPLIB 654 instance with Euclidean norm and $\bar{\mathbf{x}} = (1500, 1500)$ and different budgets

As the results in Tables 2 and 4 show, the value of the objective function of the problem with respect to the new weights $\hat{\mathbf{w}}$, i.e. $g_4(\mathbf{r}, \mathbf{s}, \bar{\mathbf{x}})$, is smaller than the value of the objective function with respect to the original weights $\mathbf{w}$, i.e. $f(\bar{\mathbf{x}}, \mathbf{w})$. Therefore, By allocating budget and adjusting weights, we can enhance the sum of weighted distances between clients and the facility. As mentioned, in problem $\mathbf{P}$, weight correction is only by reduction, i.e. $\hat{w}_i = w_i - s_i$.

Figure 1 demonstrate that increasing the budget leads to a further reduction in weight. It is important to note that the weight of the i-th point can be reduced by a maximum of $w_i$, which is equivalent to $s_i = w_i$. When this maximum reduction occurs, the objective function of the problem reaches a value of zero. Consequently, all points can be regarded as optimal solutions.

Table 5 shows the results for different norms.

## 5 Consequence

In this paper, we presented an algorithm for reverse continuous location problem with variable weights. The optimality condition has been proven for the given algorithm. We showed that the above problem can be transformed into a continuous knapsack problem. The results show the algorithm could efficiently find the optimal solutions.

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Table 5: The results of Algorithm $A_1$ for the instances from [6] with $L_p$ norms.

| Instance | $n$ | $\bar{\mathbf{x}} = (\bar{x}, \bar{y})$ | $B$ | $p$ | $\mathbf{x}_k = (x_k, y_k)$ | $g_2(\mathbf{r}, \mathbf{s}, \bar{\mathbf{x}})$ | $f(\bar{\mathbf{x}}, \mathbf{w}^{(0)})$ | $CPU$ |
|---|---|---|---|---|---|---|---|---|
| Data | 18 | $(2, 2)$ | 50 | 3 | $(4.582, 2.7436)$ | 49.311 | 184.69 | 0.0119 |
| | | | | 4 | $(4.2073, 2.9215)$ | 48.257 | 179.94 | 0.0089 |
| | | | | 7 | $(3.8717, 3.1451)$ | 46.845 | 175.18 | 0.0181 |
| | | | | 8 | $(3.8217, 3.1841)$ | 46.633 | 174.55 | 0.0135 |
| Ruspini | 75 | $(50, 50)$ | 200 | 3 | $(26.53, 71.518)$ | $7.5428e+03$ | $1.3912e+04$ | 0.0133 |
| | | | | 4 | $(25.173, 70.842)$ | $7.4136e+03$ | $1.3669e+04$ | 0.0126 |
| | | | | 7 | $(23.775, 70.229)$ | $7.306e+03$ | $1.3444e+04$ | 0.0135 |
| | | | | 8 | $(24.9557, 97.224)$ | $7.2947e+03$ | $1.3417e+04$ | 0.0116 |
| Bongartz | 287 | $(15, 35)$ | 500 | 3 | $(21.357, 30.848)$ | $4.6226e+03$ | $7.2448e+03$ | 0.0110 |
| | | | | 4 | $(21.396, 30.734)$ | $4.4808e+03$ | $7.054e+03$ | 0.0172 |
| | | | | 7 | $(21.318, 30.835)$ | $4.3414e+03$ | $6.8739e+03$ | 0.0171 |
| | | | | 8 | $(21.296, 30.862)$ | $4.324e+03$ | $6.852e+03$ | 0.0119 |
| TSPLIB | 654 | $(2000, 4000)$ | 3000 | 3 | $(1962.1, 3809)$ | $1.7551e+06$ | $4.8555e+06$ | 0.0164 |
| | | | | 4 | $(2060.6, 3815.1)$ | $1.7136e+06$ | $4.7707e+06$ | 0.0132 |
| | | | | 7 | $(2171.1, 3819.7)$ | $1.6765e+06$ | $4.7062e+06$ | 0.0132 |
| | | | | 8 | $(2178.7, 3831.5)$ | $1.6722e+06$ | $4.6998e+06$ | 0.0129 |

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