



Weak module amenability of module extension Banach algebras

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Abstract

In this paper, we study module and weak module amenability of the module extension Banach algebra $A \oplus X$ of a Banach algebra A by a Banach A -module X . As an example, we show that for an inverse semigroup S with set of idempotents E , the module extension $\ell^1(E) \oplus \ell^1(S)$ is amenable as an $\ell^1(E)$ -module if and only if S is amenable. We also study module biflatness and module biprojectivity of module extensions.

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1 Introduction

The notion of amenability for Banach algebras was first introduced by Johnson in [16]. A linear map $D : \mathcal{B} \rightarrow \mathcal{E}$, between Banach algebras, is a derivation if

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{B}).$$

A Banach algebra \mathcal{B} is amenable if every continuous derivation D from \mathcal{B} into any dual Banach \mathcal{B} -bimodule \mathcal{E}^* is inner, namely there exists $f \in \mathcal{E}^*$ such that

$$D(a) = a \cdot f - f \cdot a \quad (a \in \mathcal{B}),$$

where the module actions on \mathcal{E}^* are defined by

$$\langle f \cdot a, x \rangle = \langle f, a \cdot x \rangle \quad \text{and} \quad \langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle \quad (a \in \mathcal{B}, x \in \mathcal{E}, f \in \mathcal{E}^*).$$

In [17], Johnson proved that a Banach algebra \mathcal{B} is amenable if and only if it has a bounded approximate diagonal, that is, a bounded net $(m_a) \subseteq \widehat{\mathcal{B}} \otimes \mathcal{B}$ such that

$$a \cdot m_a - m_a \cdot a \rightarrow 0 \quad \text{and} \quad \Pi_{\mathcal{B}}(m_a)a \rightarrow a \quad (a \in \mathcal{B}),$$

where $\Pi_{\mathcal{B}} : \widehat{\mathcal{B}} \otimes \mathcal{B} \rightarrow \mathcal{B}$ is the multiplication map defined by $\Pi_{\mathcal{B}} : (a \otimes b) = ab$.

The notions of biflatness and biprojectivity for Banach algebras were introduced by Helemskii in [14]. A Banach algebra \mathcal{B} is biflat if there is a bounded \mathcal{B} -bimodule homomorphism $\rho : \mathcal{B} \rightarrow (\widehat{\mathcal{B}} \otimes \mathcal{B})^{**}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\mathcal{K}_{\mathcal{B}}} & \mathcal{B}^{**} \\ & \searrow \rho & \uparrow \pi_{\mathcal{B}}^{**} \\ & & (\widehat{\mathcal{B}} \otimes \mathcal{B})^{**} \end{array}$$

where $\mathcal{K}_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B}^{**}$ is the natural embedding of \mathcal{B} into its second dual, and we regard $\widehat{\mathcal{B}} \otimes \mathcal{B}$ as a Banach \mathcal{B} -bimodule with module actions:

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in \mathcal{B}).$$

Similarly, \mathcal{B} is biprojective if there is a bounded \mathcal{B} -bimodule homomorphism $\sigma : \mathcal{B} \rightarrow$

$(\mathcal{B} \widehat{\otimes} \mathcal{B})$ such that $\Pi_{\mathcal{B}}(\sigma) = \text{Id}$ and the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{B} & \xrightarrow{id} & \mathcal{B}^{**} \\
 & \searrow \pi_{\mathcal{B}} & \uparrow \sigma \\
 & & (\mathcal{B} \widehat{\otimes} \mathcal{B})^{**}
 \end{array}$$

It is known that \mathcal{B} is amenable if and only if \mathcal{B} is biflat and has a bounded approximate identity [14]. Biflatness and bprojectivity are studied for various classes of Banach algebras, including C^* -algebras, group algebras and Segal algebras [14, 25]. Consider the situation where \mathcal{B} has an extra module structure as a Banach module over another Banach algebra \mathcal{B} with compatible actions. The second author introduced and studied module amenability of \mathcal{B} in [1]. The module versions of weak amenability [2] permanent amenability [8] super amenability [28] contractibility [5, 6] biflatness and bprojectivity [7] topological center [3] and Arens regularity [24] are studied by several authors.

The main motivating example in most of the works cited above was $\mathcal{B} = \ell^1(S)$ and $\mathcal{B} = \ell^1(E)$, where S is an inverse semigroup with set of idempotents E . In this paper we consider another class of examples coming from module extensions of Banach algebras, in which $\mathcal{B} = A \oplus X$ and $\mathcal{B} = A$, where A is a Banach algebra, X is a Banach A -module, and $A \oplus X$ is the module extension of A by X , considered as a Banach algebra and Banach A -module (compare with [22]). This class of Banach algebras first appeared in [10, 4] and contains the class of triangular Banach algebras [18]. The amenability and weak amenability of these algebras are studied in general in [20, 21] and in the special case of triangular Banach algebras in [12, 13, 18]. The biflatness and bprojectivity of these algebras are studied in [19].

2 Module amenability

Through out this paper, \mathcal{A} and \mathfrak{A} are Banach algebras such that \mathcal{A} is Banach \mathfrak{A} -bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, (ab) \cdot \alpha = a(b \cdot \alpha) \quad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

We say that \mathfrak{A} acts trivially on \mathcal{A} from left if $\alpha \cdot a = f(\alpha)a$, for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, where $f \in \Phi_{\mathfrak{A}}$ is a character on \mathfrak{A} .

Let \mathcal{X} be a Banach \mathcal{A} -bimodule and a Banach \mathfrak{A} -bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a) \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in \mathcal{X}),$$

and the same for the right or two-sided actions. Then we say that \mathcal{X} is a Banach \mathcal{A} - \mathfrak{A} -module. If moreover

$$\alpha \cdot x = x \cdot \alpha \quad (\alpha \in \mathfrak{A}, x \in \mathcal{X}),$$

\mathcal{X} is called a commutative \mathcal{A} - \mathfrak{A} -module.

If \mathcal{X} is a commutative Banach \mathcal{A} - \mathfrak{A} -module, then so is \mathcal{X}^* , where the actions of \mathcal{A} and \mathfrak{A} and \mathcal{X}^* are defined by

$$\langle \alpha \cdot f, x \rangle = \langle f, x \cdot \alpha \rangle, \langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in \mathcal{X}, f \in \mathcal{X}^*),$$

and the same for the right actions. Let \mathcal{Y} be another \mathcal{A} - \mathfrak{A} -module then a \mathcal{A} - \mathfrak{A} -module morphism from \mathcal{X} to \mathcal{Y} is a norm-continuous map $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ with $\phi(x \pm y) = \phi(x) \pm \phi(y)$ and

$$\phi(\alpha \cdot x) = \alpha \cdot \phi(x), \phi(x \cdot \alpha) = \phi(x) \cdot \alpha, \phi(a \cdot x) = a \cdot \phi(x), \phi(x \cdot a) = \phi(x) \cdot a,$$

for $x, y \in \mathcal{X}, a \in \mathcal{A}$ and \mathfrak{A} . Consider the projective tensor product $\mathcal{A} \hat{\otimes} \mathcal{A}$, which is a Banach algebra with respect to the canonical multiplication

$$(a \otimes b)(c \otimes d) = ac \otimes bd,$$

and extended by linearity and continuity [9]. Also $\mathcal{A} \hat{\otimes} \mathcal{A}$ is a Banach \mathcal{A} - \mathfrak{A} -module with canonical actions. Let I be the closed ideal of $\mathcal{A} \hat{\otimes} \mathcal{A}$ generated by elements of the form $(a \cdot \alpha) \otimes b - a \otimes (\alpha \cdot b)$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$. Consider the multiplication map $\pi_{\mathcal{A}} : \mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$ defined by $\pi_{\mathcal{A}}(a \otimes b) = ab$, extended by linearity and continuity. Let $J_{\mathcal{A}}$ be the closed ideal of \mathcal{A} generated by $\pi_{\mathcal{A}}(I)$. Then the module projective tensor product $\mathcal{A} \hat{\otimes}_{\mathfrak{A}} \mathcal{A} \cong \frac{\mathcal{A} \hat{\otimes} \mathcal{A}}{I}$ and the quotient Banach algebra $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ are Banach \mathfrak{A} -modules with compatible actions. Also the map $\tilde{\pi}_{\mathcal{A}} : \mathcal{A} \hat{\otimes}_{\mathfrak{A}} \mathcal{A} \rightarrow \frac{\mathcal{A}}{J_{\mathcal{A}}}$ defined by $\tilde{\pi}_{\mathcal{A}}(a \otimes b + I) = ab + I$ extends to an \mathfrak{A} -module morphism. Let \mathcal{A} and \mathfrak{A} be as above and \mathcal{X} be a Banach \mathcal{A} - \mathfrak{A} -module. A bounded \mathcal{A} - \mathfrak{A} -module morphism $D : \mathcal{A} \rightarrow \mathcal{X}$ is called a module derivation if

$$D(a \pm b) = D(a) \pm D(b) \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

When \mathcal{X} is commutative, each $x \in \mathcal{X}$ defines a module derivation

$$D_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

These are called inner module derivations. The Banach algebra \mathcal{A} is called module amenable (as an \mathfrak{A} -module) if for any commutative Banach \mathcal{A} - \mathfrak{A} -module \mathcal{X} , each module derivation $D : \mathcal{A} \rightarrow \mathcal{X}^*$ is inner [1]. The Banach algebra \mathcal{A} is called weakly module amenable (as an \mathfrak{A} -module) if $(\frac{\mathcal{A}}{J_{\mathcal{A}}})^* = J_{\mathcal{A}}^1$ is commutative Banach \mathfrak{A} -module and each module derivation from \mathcal{A} to $(\frac{\mathcal{A}}{J_{\mathcal{A}}})^* = J_{\mathcal{A}}^1$ is inner [2]. For a Banach module \mathcal{X} over \mathcal{A} , a net (a_α) in \mathcal{A} is called a bounded approximate identity for \mathcal{X} if

$$\| a_\alpha \cdot x - x \| + \| x \cdot a_\alpha \| \rightarrow 0 \quad (x \in \mathcal{X}).$$

The following results are proved in [3].

Theorem 2.1. If \mathfrak{A} has bounded approximate identity for \mathcal{A} , then amenability of $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ implies module amenability of \mathcal{A} . Conversely if \mathcal{A} is module amenable as an \mathfrak{A} -module with trivial left action, J_0 is a closed ideal of \mathcal{A} such that $J_{\mathcal{A}} \subseteq J_0$ and $\frac{\mathcal{A}}{J_0}$ has a left bounded approximate identity, then $\frac{\mathcal{A}}{J_0}$ is amenable.

Theorem 2.2. Let \mathfrak{A} acts trivially on \mathcal{A} from left and $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ has a left or right bounded approximate identity, then weak module amenability of \mathcal{A} implies weak amenability of $\frac{\mathcal{A}}{J_{\mathcal{A}}}$. The converse is true if \mathcal{A} is a right essential \mathfrak{A} -module.

3 Module extension Banach algebras

In this section we study module extensions of a Banach algebra \mathcal{A} by a Banach \mathcal{A} -bimodule \mathcal{X} . This is the Banach algebra $A \oplus \mathcal{X}$, the ℓ^1 -direct sum of \mathcal{A} and \mathcal{X} , with the algebra product

$$(a, x).(b, y) = (ab, a.y + x.b), \quad (a, b \in A, x, y \in X).$$

We obtain the necessary and sufficient conditions for a module extension Banach algebra to be module amenable, weakly module amenable, module biflat, or module biprojective, as an \mathfrak{A} -module.

We consider the Banach algebra \mathcal{A} as an \mathcal{A} -bimodule with the following compatible module actions

$$b.a = ba, \quad a.b = f(a, b) \quad (a, b \in \mathcal{A}, f \in \Phi_{\mathcal{A}}),$$

where $\Phi_{\mathcal{A}}$ is the character space of \mathcal{A} . Then $J_{\mathcal{A}}$ is the closed ideal of \mathcal{A} generated by the set $\{bac - f(a)bc : a, b, c \in \mathcal{A}\}$. Consider the module extension $\mathcal{B} := \mathcal{A} \oplus \mathcal{X}$ as an \mathcal{A} -bimodule with the following compatible module actions

$$(b, x).a = (ba, xa), \quad a.(b, x) = (f(a)b, f(a)x) \quad (a, b \in \mathcal{A}, x \in X, f \in \Phi_{\mathcal{A}}),$$

then

$$\begin{aligned} J_{\mathcal{B}} &= \{[(b, x).a](c, y) - (bx)[a.(c, y)] : a, b, c \in \mathcal{A}, x, y \in \mathcal{X}\} \\ &= \{(ba, xa)(c, y) - (bx)(f(a)c, f(a)y) : a, b, c \in \mathcal{A}, x, y \in \mathcal{X}\} \\ &= \{(bac - bf(a)c, (ba - f(a)b)y + x(ac - f(a)c)) : a, b, c \in \mathcal{A}, x, y \in \mathcal{X}\} \\ &= J_{\mathcal{A}} \oplus (\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}) = J_{\mathcal{A}} \oplus \mathcal{X}, \end{aligned}$$

when \mathcal{X} is an essential \mathcal{A} -bimodule.

Proposition 3.1. If $\{e_\lambda\}$ is a bounded approximate identity for \mathcal{A} then $\{e_\alpha\}$ is a bounded approximate identity for the module extension $\mathcal{A} \oplus \mathcal{X}$ when \mathcal{X} is an essential \mathcal{A} -bimodule.

Proof . For $(a, x) \in \mathcal{A} \oplus \mathcal{X}$,

$$\begin{aligned} \|e_\alpha \cdot (a, x) - (a, x)\| &= \|f(e_\alpha a, f(e_\alpha)x)(a, x)\| \\ &= \|f(e_\alpha) - 1\| \|(a, x)\| \longrightarrow 0, \end{aligned}$$

and if $x = y.b$, for some $b \in A$ and $y \in \mathcal{X}$, then

$$\begin{aligned} \|(a, x).e_\alpha - (a, x)\| &= \|(ae_\alpha, x.e_\alpha)(a, x)\| \\ &= \|ae_\alpha - a\| + \|x.e_\alpha - x\| \\ &= \|ae_\alpha - a\| + \|(y.b).e_\alpha - (y.b)\| \\ &\leq \|ae_\alpha - a\| + \|y\| \|be_\alpha - b\| \longrightarrow 0. \end{aligned}$$

□

Theorem 3.2. Let \mathcal{A} has a bounded approximate identity and \mathcal{X} is an essential \mathcal{A} -bimodule. Then module extension Banach algebra $\mathcal{A} \oplus \mathcal{X}$ is module amenable as \mathcal{A} -bimodule if and only if the Banach algebra $\frac{\mathcal{A}}{J_A}$ is amenable.

Proof . We define the map $\varphi := \mathcal{A} \oplus \mathcal{X} \longrightarrow \frac{\mathcal{A}}{J_A}$ by $\varphi((a, x)) = a + J_A$. It is easy to see that φ is a well-defined \mathcal{A} -module morphism and an algebra homomorphism and

$$\ker \varphi = \{(a, x) : a \in J_A, x \in X = J_A \oplus X\}$$

and we have $\frac{\mathcal{A} \oplus \mathcal{X}}{J_{\mathcal{A} \oplus \mathcal{X}}} \simeq \frac{\mathcal{A}}{J_A}$, that is $\frac{\mathcal{A} \oplus \mathcal{X}}{J_B} \simeq \frac{\mathcal{A}}{J_A}$. Therefore, by Theorem 2.1 we get the result.

□

Theorem 3.3. Let \mathcal{A} has a bounded approximate identity for itself and for \mathcal{X} . Then the module extension Banach algebra $\mathcal{A} \oplus \mathcal{X}$ is weakly module amenable as an \mathcal{A} -bimodule if and only if the Banach algebra $\frac{\mathcal{A}}{J_A}$ is weakly amenable.

Proof . This follows from Theorem 2.3 and the fact that $\frac{\mathcal{A} \oplus \mathcal{X}}{J_B} \simeq \frac{\mathcal{A}}{J_A}$. □

As an example, let S is an inverse semigroup with an upward directed set of idempotents E , then E satisfies condition D_1 of Duncan and Namoika [11], hence $\ell^1(E)$ has a bounded approximate identity. If $\{g_j\}$ is a bounded approximate identity of $\ell^1(E)$, then

$$g_j * \delta_s = g_j * \delta_{ss^*s} = g_j * \delta_{ss^*} * \delta_s \longrightarrow \delta_s \quad (s \in S),$$

and similarly for the right multiplication. Therefore $\ell^1(E)$ has a bounded approximate identity for $\ell^1(S)$. Consider $A = \ell^1(E)$, $X = \ell^1(S)$ and let $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left, that is

$$\delta_e \cdot \delta_s = \delta_s, \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e \quad (s \in S, e \in E).$$

It is easy to show that $\ell^1(S)$ is a Banach $\ell^1(E)$ -module with compatible actions. Define an equivalence relation on E as follows:

$$e_1 \approx e_2 \iff \delta_{e_1} - \delta_{e_2} \in J_{\ell^1(E)} \quad (e_1, e_2 \in E).$$

Then the quotient $\frac{E}{\approx}$ is discrete group (see [1, 2]). As in [2], one may observe that $\frac{\ell^1(E)}{J_{\ell^1(E)}} \cong \ell^1(\frac{E}{\approx})$. Thus by Theorems 3.2 and 3.3, the module extension $\mathcal{B} = \ell^1(E) \oplus \ell^1(S)$ is module amenable as an $\ell^1(E)$ -bimodule if and only if the discrete group $\frac{E}{\approx}$ is amenable. Also it is always weakly module amenable as an $\ell^1(E)$ -bimodule, since the group algebra $\ell^1(\frac{E}{\approx})$ is always weakly amenable.

As a negative result, consider the case $\mathcal{A} = \mathcal{C}$, $\mathcal{X} = \mathcal{C}$ and let \mathcal{A} act on \mathcal{X} by multiplication from both sides. Then the module extension $\mathcal{B} = \mathcal{C} \oplus \mathcal{C}$ is module amenable as an \mathcal{C} -bimodule by Theorem 3.2 (since $\frac{\mathcal{C}}{J_{\mathcal{A}}} \simeq \mathcal{C}$ is amenable), but it is not even weak amenable [26].

Next we turn to module biflatness and module biprojectivity of module extension Banach algebras. First let us recall some definition and results from [7].

Definition 3.4. The Banach algebra \mathcal{A} is called module biprojective (as an \mathfrak{A} -module) if $\tilde{\Pi}_{\mathcal{A}} : \mathcal{A} \hat{\otimes}_{\mathfrak{A}} \mathcal{A} \longrightarrow \frac{\mathcal{A}}{J_{\mathcal{A}}}$ has a bounded right inverse which is an $\frac{\mathcal{A}}{J_{\mathcal{A}}} - \mathfrak{A} - module$ morphism. It is called module biflat (as an \mathfrak{A} -module) if $\tilde{\Pi}_{\mathcal{A}}^* : (\frac{\mathcal{A}}{J_{\mathcal{A}}})^* \longrightarrow (\mathcal{A} \hat{\otimes}_{\mathfrak{A}} \mathcal{A})^*$ has a bounded left inverse which is an $\frac{\mathcal{A}}{J_{\mathcal{A}}} - \mathfrak{A}$ -module morphism.

Proposition 3.5. Assume that \mathfrak{A} acts on \mathcal{A} trivially from left and $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ has a left identity and \mathfrak{A} has a bounded approximate identity for \mathcal{A} . If \mathcal{A} is module (biflat) biprojective, then $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ is (biflat) biprojective.

Now let \mathcal{X} be an essential \mathcal{A} -bimodule and consider $\mathcal{B} = \mathcal{A} \oplus \mathcal{X}$ as an \mathcal{A} -bimodule with trivially left action and canonical right action.

Similar to the proof of Theorem 3.2, and using the above proposition, we get the following result.

Corollary 3.6. Let \mathcal{A} has a bounded approximate identity and $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ has a left identity. If the module extension $\mathcal{A} = \mathcal{A} \oplus \mathcal{X}$ is module (biflat) biprojective then $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ is (biflat) biprojective.

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