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Fuzzy HUR-stability of an GCJ functional equation: A fixed point alternative approach

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Abstract

Using the fixed point method, we prove the generalized Hyers-Ulam(or Hyers-Ulam-Rassias) stability of the following generalized Cauchy-Jensen functional equation

$$2f\left(\frac{\sum_{i=1}^{p} x_i+\sum_{j=1}^{q} y_j}{2}+\sum_{k=1}^{d} z_k\right)=\sum_{i=1}^{p} f(x_i)+\sum_{j=1}^{q} f(y_j)+2\sum_{k=1}^{d} f(z_k)$$

where p, q and d are positive natural numbers greater than 1, in fuzzy Banach spaces. The concept of Hyers-Ulam-Rassias stability originated from Th. M. Rassias stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), 297-300.

Keywords: Hyers-Ulam-Rassias stability, Fuzzy Banach space, Generalized Cauchy-Jensen functional equation, Fixed point method.

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1 Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [39] concerning the stability of group homomorphisms. Hyers [21] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Th. M. Rassias [30] for linear mappings by considering an unbounded Cauchy difference.
The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The Hyers-Ulam stability of the quadratic functional equation was proved by Skof [38] for mappings $f: X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [9] proved the Hyers-Ulam stability of the quadratic functional equation.

In this paper, we consider the following *generalized Cauchy-Jensen functional equation*

$$2f\left(\frac{\sum_{i=1}^{p} x_i + \sum_{j=1}^{q} y}{2} + \sum_{k=1}^{d} z_k\right) = \sum_{i=1}^{p} f(x_i) + \sum_{j=1}^{q} f(y_j) + 2\sum_{k=1}^{d} f(z_k) \tag{1.1}$$

and prove the generalized Hyers-Ulam stability of the functional equation (1.1) in fuzzy Banach spaces. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [1]–[3], [6]– [18],[28]–[35]).

Katsaras [22] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms a vector space from various points of view (see [19, 24, 28]). In particular, Bag and Samanta [4], following Cheng and Mordeson [7], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [23]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [5].

Definition 1.1. (Bag and Samanta [4]) Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,
$(N1)$ $N(x,t) = 0$ for $t \leq 0$;
$(N2)$ $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
$(N3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;
$(N4)$ $N(x+y, c+t) \geq min\{N(x,s), N(y,t)\}$;

$(N5)$ $N(x,.)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
$(N6)$ for $x \neq 0$, $N(x,.)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed vector space.

Example 1.1. Let $(X, \|.\|)$ be a normed linear space and $\alpha, \beta > 0$. Then

$$N(x,t) = \begin{cases} \frac{\alpha t}{\alpha t+\beta\|x\|} & t > 0, x \in X \\ 0 & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.2. (Bag and Samanta [4]) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{t\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X and we denote it by $N - \lim_{t\to\infty} x_n = x$.

Definition 1.3. (Bag and Samanta [4]) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.
We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [5]).
Throughout this paper, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

Definition 1.4. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:
(1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1. ([6, 10]) Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty$ for all $n_0 \geq n_0$ and all $x \in X$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$;

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

2 Fuzzy stability of the functional equation (1.1)

Throughout this paper, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

Theorem 2.1. Let $\varphi : X^{p+q+d} \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with

$$\varphi\left(\frac{2x_i}{p+q+2d}, \frac{2y_j}{p+q+2d}, \frac{2z_k}{p+q+2d}\right) \leq \frac{2\alpha}{p+q+2d}\varphi(x_i, y_j, z_k) \tag{2.1}$$

for all $x_i, y_j, z_k \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &N\left(2f\left(\frac{\sum_{i=1}^{p} x_i + \sum_{j=1}^{q} y}{2} + \sum_{k=1}^{d} z_k\right) - \sum_{i=1}^{p} f(x_i) - \sum_{j=1}^{q} f(y_j) - 2\sum_{k=1}^{d} f(z_k), t\right) \\ &\geq \frac{t}{t + \varphi(x_i, y_j, z_k)} \end{aligned} \tag{2.2}$$

for all $x_i, y_j, z_k \in X$ and all $t > 0$. Then the limit $L(x) := N\text{-}\lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n x}{(p+q+2d)^n}\right)$ exists for each $x \in X$ and defines a unique generalized Cauchy-Jensen mapping $L : X \to Y$ such that

$$N(f(x) - L(x), t) \geq \frac{(p+q+2d)(1-\alpha)t}{(p+q+2d)(1-\alpha)t + \alpha\varphi(x, x, \cdots, x)}. \tag{2.3}$$

Proof . Putting $x_i = y_j = z_k = x$ in (2.2), we have

$$N\left(2f\left(\frac{(p+q+2d)x}{2}\right) - (p+q+2d)f(x), t\right) \geq \frac{t}{t + \varphi(\underbrace{x, x, \cdots, x}_{(p+q+d)-times})} \tag{2.4}$$

for all $x \in X$ and $t > 0$. Replacing x by $\frac{2x}{p+q+2d}$ in (2.4), we obtain

$$\begin{aligned} N\left(f(x) - \frac{p+q+2d}{2} f\left(\frac{2x}{p+q+2d}\right), \frac{t}{2}\right) \quad &\geq \quad \frac{t}{t + \varphi\left(\frac{2x}{p+q+2d}, \frac{2x}{p+q+2d}, \cdots, \frac{2x}{p+q+2d}\right)} \\ &\geq \quad \frac{t}{t + \frac{2\alpha}{p+q+2d}\varphi(x, x, \cdots, x)}. \end{aligned} \tag{2.5}$$

for all $x \in X$ and $t > 0$. So

$$N\left(f(x) - \frac{p+q+2d}{2} f\left(\frac{2x}{p+q+2d}\right), \frac{\alpha t}{p+q+2d}\right) \geq \frac{t}{t + \varphi(\underbrace{x, x, \cdots, x}_{(p+q+d)-times})} \tag{2.6}$$

Consider the set $S := \{g : X \to Y, g(0) = 0\}$ and the generalized metric d in S defined by

$$d(f, g) = \inf\left\{\mu \in \mathbb{R}^+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, x, \cdots, x)}, \forall x \in X,\ t > 0\right\},$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [25, Lemma 2.1]).

Now, we consider a linear mapping $J : S \to S$ such that $Jg(x) := \frac{p+q+2d}{2} g\left(\frac{2x}{p+q+2d}\right)$ for all $x \in X$. Let $g, h \in S$ satisfy $d(g, h) = \epsilon$. Then

$$N(g(x) - h(x), \epsilon t) \geq \frac{t}{t + \varphi(x, x, \cdots, x)}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), \alpha\epsilon t) &= N\left(\frac{p+q+2d}{2} g\left(\frac{2x}{p+q+2d}\right) - \frac{p+q+2d}{2} h\left(\frac{2x}{p+q+2d}\right), \alpha\epsilon t\right) \\ &= N\left(g\left(\frac{2x}{p+q+2d}\right) - h\left(\frac{2x}{p+q+2d}\right), \frac{2\alpha\epsilon t}{p+q+2d}\right) \\ &\geq \frac{\frac{2\alpha t}{p+q+2d}}{\frac{2\alpha t}{p+q+2d} + \varphi\left(\frac{2x}{p+q+2d}, \frac{2x}{p+q+2d}, \cdots, \frac{2x}{p+q+2d}\right)} \\ &\geq \frac{\frac{2\alpha t}{p+q+2d}}{\frac{2\alpha t}{p+q+2d} + \frac{2\alpha}{p+q+2d}\varphi(x, x, \cdots, x)} = \frac{t}{t + \varphi(x, x, \cdots, x)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq \alpha\epsilon$. This means that $d(Jg, Jh) \leq \alpha d(g, h)$ for all $g, h \in S$. It follows from (2.6) that $d(f, Jf) \leq \frac{\alpha}{p+q+2d}$. By Theorem 1.1, there exists a mapping $L : X \to Y$ satisfying the following:

(1) L is a fixed point of J, that is,

$$L\left(\frac{2x}{p+q+2d}\right) = \frac{2}{p+q+2d} L(x) \tag{2.7}$$

for all $x \in X$. The mapping L is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that L is a unique mapping satisfying (2.7) such that there exists $\mu \in (0, \infty)$ satisfying $N(f(x) - L(x), \mu t) \geq \frac{t}{t+\varphi(x,x,\cdots,x)}$ for all $x \in X$ and $t > 0$.

(2) $d(J^n f, L) \to 0$ as $n \to \infty$. This implies the equality $\lim_{n\to\infty} N\text{-}\lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n x}{(p+q+2d)^n}\right) = L(x)$ for all $x \in X$.

(3) $d(f, L) \leq \frac{d(f,Jf)}{1-\alpha}$ with $f \in \Omega$, which implies the inequality $d(f, L) \leq \frac{\alpha}{(p+q+2d)(1-\alpha)}$. This implies that the inequality (2.3) holds.

Replacing x_i, y_j and z_k by $\frac{2^n x_i}{(p+q+2d)^n}$, $\frac{2^n y_j}{(p+q+2d)^n}$ and $\frac{2^n z_k}{(p+q+2d)^n}$, respectively, in (2.2), we get

$$\begin{aligned}&N\Bigg(\frac{(p+q+2d)^n}{2^n}\Bigg[2f\left(\frac{\sum_{i=1}^{p}\frac{2^n x_i}{(p+q+2d)^n}+\sum_{j=1}^{q}\frac{2^n y_j}{(p+q+2d)^n}}{2}+\sum_{k=1}^{d}\frac{2^n z_k}{(p+q+2d)^n}\right)\\&-\sum_{i=1}^{p}f\left(\frac{2^n x_i}{(p+q+2d)^n}\right)-\sum_{j=1}^{q}f\left(\frac{2^n y_j}{(p+q+2d)^n}\right)\\&-\sum_{k=1}^{d}f\left(\frac{2^n z_k}{(p+q+2d)^n}\right)\Bigg],\frac{(p+q+2d)^n t}{2^n}\Bigg)\geq\frac{t}{t+\varphi\Big(\frac{2^n x_i}{(p+q+2d)^n},\frac{2^n y_j}{(p+q+2d)^n},\frac{2^n z_k}{(p+q+2d)^n}\Big)}\end{aligned}\tag{2.8}$$

for all $x_i, y_j, z_k \in X$, $t > 0$ and all $n \in \mathbb{N}$. So by (2.1) and (2.8), we have

$$\begin{aligned}&N\Bigg(\frac{(p+q+2d)^n}{2^n}\Bigg[2f\left(\frac{\sum_{i=1}^{p}\frac{2^n x_i}{(p+q+2d)^n}+\sum_{j=1}^{q}\frac{2^n y_j}{(p+q+2d)^n}}{2}+\sum_{k=1}^{d}\frac{2^n z_k}{(p+q+2d)^n}\right)\\&-\sum_{i=1}^{p}f\left(\frac{2^n x_i}{(p+q+2d)^n}\right)-\sum_{j=1}^{q}f\left(\frac{2^n y_j}{(p+q+2d)^n}\right)\\&-\sum_{k=1}^{d}f\left(\frac{2^n z_k}{(p+q+2d)^n}\right)\Bigg],t\Bigg)\geq\frac{\frac{2^n t}{(p+q+2d)^n}}{\frac{2^n t}{(p+q+2d)^n}+\frac{2^n\alpha^n t}{(p+q+2d)^n}\varphi(x_i,y_j,z_k)}.\end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{\frac{2^n t}{(p+q+2d)^n}}{\frac{2^n t}{(p+q+2d)^n}+\frac{2^n\alpha^n t}{(p+q+2d)^n}\varphi(x_i,y_j,z_k)}=1$$

for all $x_i, y_j, z_k \in X$ and all $t > 0$, we deduce that

$$N\left(2L\left(\frac{\sum_{i=1}^{p}x_i+\sum_{j=1}^{q}y}{2}+\sum_{k=1}^{d}z_k\right)-\sum_{i=1}^{p}L(x_i)-\sum_{j=1}^{q}L(y_j)-2\sum_{k=1}^{d}L(z_k),t\right)=1$$

for all $x_i, y_j, z_k \in X$ and all $t > 0$. Thus the mapping $L : X \to Y$ satisfying (1.1), as desired. This completes the proof. □

Corollary 2.1. Let $\theta \geq 0$ and let r be a real number with $r > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned}&N\left(2f\left(\frac{\sum_{i=1}^{p}x_i+\sum_{j=1}^{q}y}{2}+\sum_{k=1}^{d}z_k\right)-\sum_{i=1}^{p}f(x_i)-\sum_{j=1}^{q}f(y_j)-2\sum_{k=1}^{d}f(z_k),t\right)\\&\geq\frac{t}{t+\theta\left(\sum_{i=1}^{p}\|x_i\|^r+\sum_{j=1}^{q}\|y_j\|^r+\sum_{k=1}^{d}\|z_k\|^r\right)}\end{aligned}\tag{2.9}$$

for all $x_i, y_j, z_k \in X$ and all $t > 0$. Then $L(x) := N\text{-}\lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n x}{(p+q+2d)^n}\right)$ exists for each $x \in X$ and defines a generalized Cauchy-Jensen mapping $L : X \to Y$ such that

$$N(f(x)-L(x),t) \geq \frac{((p+q+2d)-2^{r-1}(p+q+2d)^{2-r})t}{((p+q+2d)-2^{r-1}(p+q+2d)^{2-r})t + 2^{r-1}(p+q+d)(p+q+2d)^{1-r}\theta\|x\|^r}$$

for all $x \in X$ and all $t > 0$.

Proof . The proof follows from Theorem 2.1 by taking

$$\varphi(x_i, y_j, z_k) := \theta\left(\sum_{i=1}^{p}\|x_i\|^r + \sum_{j=1}^{q}\|y_j\|^r + \sum_{k=1}^{d}\|z_k\|^r\right)$$

for all $x_i, y_j, z_k \in X$. Then we can choose $\alpha = \left(\frac{p+q+2d}{2}\right)^{1-r}$ and we get the desired result. □

Theorem 2.2. Let $\varphi : X^{p+q+d} \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with

$$\varphi\left(\frac{(p+q+2d)x_i}{2}, \frac{(p+q+2d)y_j}{2}, \frac{(p+q+2d)z_k}{2}\right) \leq \frac{(p+q+2d)\alpha}{2}\varphi(x_i, y_j, z_k) \qquad (2.10)$$

for all $x_i, y_j, z_k \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then the limit

$$R(x) := N\text{-}\lim_{n\to\infty} \frac{2^n}{(p+q+2d)^n} f\left(\frac{(p+q+2d)^n x}{2^n}\right)$$

exists for each $x \in X$ and defines a generalized Cauchy-Jensen mapping $R : X \to Y$ such that

$$N(f(x)-R(x),t) \geq \frac{(p+q+2d)(1-\alpha)t}{(p+q+2d)(1-\alpha)t + \varphi(x,x,\cdots,x)} \qquad (2.11)$$

Proof . Let (S, d) be the generalized metric space defined as in the proof of Theorem 2.1. Consider the linear mapping $J : S \to S$ such that $Jg(x) := \frac{2}{p+q+2d} g\left(\frac{(p+q+2d)x}{2}\right)$ for all $x \in X$. Let $g, h \in S$ be such that $d(g,h) = \epsilon$. Then

$$N(g(x)-h(x), \epsilon t) \geq \frac{t}{t + \varphi(x,x,\cdots,x)}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned}
&N(Jg(x) - Jh(x), \alpha\epsilon t)\\
&= N\left(\frac{2}{p+q+2d} g\left(\frac{(p+q+2d)x}{2}\right) - \frac{2}{p+q+2d} h\left(\frac{(p+q+2d)x}{2}\right), \alpha\epsilon t\right)\\
&= N\left(g\left(\frac{(p+q+2d)x}{2}\right) - h\left(\frac{(p+q+2d)x}{2}\right), \frac{(p+q+2d)\alpha\epsilon t}{2}\right)\\
&\geq \frac{\frac{(p+q+2d)\alpha t}{2}}{\frac{(p+q+2d)\alpha t}{2} + \varphi\left(\frac{(p+q+2d)x}{2}, \frac{(p+q+2d)x}{2}, \cdots, \frac{(p+q+2d)x}{2}\right)}\\
&\geq \frac{\frac{(p+q+2d)\alpha t}{2}}{\frac{(p+q+2d)\alpha t}{2} + \frac{(p+q+2d)\alpha}{2}\varphi(x, x, \cdots, x)} = \frac{t}{t + \varphi(x, x, \cdots, x)}
\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq \alpha\epsilon$. This means that $d(Jg, Jh) \leq \alpha d(g, h)$ for all $g, h \in S$. It follows from (2.4) that

$$N\left(\frac{2}{p+q+2d} f\left(\frac{(p+q+2d)x}{2}\right) - f(x), \frac{t}{p+q+2d}\right) \geq \frac{t}{t + \varphi(\underbrace{x, x, \cdots, x}_{(p+q+d)-times})}$$

for all $x \in X$ and $t > 0$. So $d(f, Jf) \leq \frac{1}{p+q+2d}$. By Theorem 1.1, there exists a mapping $R : X \to Y$ satisfying the following:
(1) R is a fixed point of J, that is,

$$\frac{p+q+2d}{2} R(x) = R\left(\frac{(p+q+2d)x}{2}\right) \tag{2.12}$$

for all $x \in X$. The mapping R is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that R is a unique mapping satisfying (2.12) such that there exists $\mu \in (0, \infty)$ satisfying $N(f(x) - R(x), \mu t) \geq \frac{t}{t+\varphi(x,x,\cdots,x)}$ for all $x \in X$ and $t > 0$.
(2) $d(J^n f, R) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} N\text{-}\lim_{n\to\infty} \frac{2^n}{(p+q+2d)^n} f\left(\frac{(p+q+2d)^n x}{2^n}\right) = R(x)$$

for all $x \in X$.
(3) $d(f, R) \leq \frac{d(f,Jf)}{1-\alpha}$ with $f \in \Omega$, which implies the inequality $d(f, R) \leq \frac{1}{(p+q+2d)(1-\alpha)}$. This implies that the inequality (2.11) holds.
The rest of the proof is similar to that of the proof of Theorem 2.1. □

Corollary 2.2. Let $\theta \geq 0$ and let r be a real number with $0 < r < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.9).

Then the limit $R(x) := N\text{-}\lim_{n\to\infty} \frac{2^n}{(p+q+2d)^n} f\left(\frac{(p+q+2d)^n x}{2^n}\right)$ exists for each $x \in X$ and defines a generalized Cauchy-Jensen mapping $R : X \to Y$ such that

$$N(f(x) - R(x), t) \geq \frac{((p+q+2d) - 2^r(p+q+2d)^{1-r})t}{((p+q+2d) - 2^r(p+q+2d)^{1-r})t + (p+q+d)\theta\|x\|^r}.$$

for all $x \in X$ and all $t > 0$.

Proof . The proof follows from Theorem 2.2 by taking

$$\varphi(x_i, y_j, z_k) := \theta\left(\sum_{i=1}^{p}\|x_i\|^r + \sum_{j=1}^{q}\|y_j\|^r + \sum_{k=1}^{d}\|z_k\|^r\right)$$

for all $x_i, y_j, z_k \in X$. Then we can choose $\alpha = \left(\frac{2}{p+q+2d}\right)^r$ and we get the desired result. □

3 Conclusion

We linked here three different disciplines, namely, the fuzzy Banach spaces, functional equations and fixed point theory. We established the Hyers-Ulam-Rassias stability of the functional equation (1.1) in fuzzy Banach spaces.

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