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On the regularity of the pointfree version of the ring of continuous functions with a countable image

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Abstract

In this paper, we introduce a new pointfree version of the ring $C_c(X)$, denoted by $C_c(L)$. This construction is defined via

$$R_\alpha = \{r \in \mathbb{R} : \operatorname{coz}(\alpha - r) \neq \top\}, \quad \text{for } \alpha \in \mathcal{R}(L),$$

where $\operatorname{coz}(\cdot)$ denotes the cozero element. Additionally, we introduce the concept of CP-frames, examine their relationship with P-frames, and provide alternative characterizations of CP-frames. We also present several properties and characterizations of CP-frames.

1 Introduction

It is well-known that $C(X)$ denotes the ring of all real-valued, continuous functions on a topological space X. Given that $C_c(X) = \{f \in C(X) : f(X) \text{ is countable}\}$ is one of

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the important subrings of $C(X)$, the studying pointfree version of $C_c(X)$ is useful. Such a study was conducted by Estaji et al., who introduced $\mathcal{R}_c(L)$ and studied the localization concepts of $C_c(X)$ [7, 14]. In particular, as in [13], the $\mathbb{R}$-subalgebras $L_c(X)$, $L_f(X)$, and $L_1(X)$ of $C(X)$ include functions with locally countable, locally finite and locally constant images, respectively. To learn more about recent studies on $L_c(X)$, we refer the reader to [5, 17]. In 2020, Estaji et al. [9] introduced R_α for the first time. Also, in 2021, Aliabad and Mahmoudi [1] studied $pim(\alpha)$ and proved that $pim(\alpha) = R_\alpha$.

Now, we use the definition of R_α to introduce another pointfree version of $C_c(X)$ via the ranges of functions, which we denote by $C_c(L)$.

2 On z_c-ideals and prime ideals in the ring $C_c(L)$

As in [9], for every $\alpha \in \mathcal{R}(L)$,

$$R_\alpha = \{r \in \mathbb{R}\colon \ \mathrm{coz}(\alpha - r) \neq \top\}.$$

In the source [1], Aliabad and Mahmoudi studied, for each $\alpha \in \mathcal{R}(L)$,

$$\mathrm{pim}(\alpha) := \bigcap \{\omega \in \mathcal{O}(\mathbb{R})\colon \alpha(\omega) = \top\}$$

and showed that $\mathrm{pim}(\alpha) = R_\alpha$.

According to [1, Proposition 2.26], we have:

(1) $R_{\alpha+\beta} \subseteq R_\alpha + R_\beta$
(2) $R_{\alpha\beta} \subseteq R_\alpha R_\beta$
(3) $R_{\alpha\vee\beta} \subseteq R_\alpha \vee R_\beta$
(4) $R_{\alpha\wedge\beta} \subseteq R_\alpha \wedge R_\beta$

where, for each $\diamond \in \{+, \cdot, \vee, \wedge\}$,

$$R_\alpha \diamond R_\beta := \left\{a \diamond b\colon a \in R_\alpha,\ b \in R_\beta\right\}.$$

Now, using the definition of R_α, we define:

$$C_c(L) := \{\alpha \in \mathcal{R}(L)\colon R_\alpha \text{ is countable}\}$$

Based on the explained materials, it is clear that, for an arbitrary frame L, $C_c(L)$ is an f-subalgebra of $\mathcal{R}(L)$.

First, we present the following definitions similar to the previous pointfree topology version of images of real-valued continuous functions with countable images. Let L be a frame.

- An ideal I in $C_c(L)$ is called a z_c-ideal if for every $\alpha \in C_c(L)$ and $\beta \in I$, $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ implies $\alpha \in I$. It is worth recalling that for every subring R of $C(X)$, the notion of z_R was introduced in [2], which provides a general framework for this concept.
- For every $a \in L$, we let $\mathbf{M}^c_a := \{\alpha \in C_c(L) : \mathrm{coz}(\alpha) \leq a\}$.
- $Coz_c[L] := \{\mathrm{coz}(\alpha) : \alpha \in C_c(L)\}$ and for every $A \subseteq C_c(L)$, we write $Coz_c[A] = \{\mathrm{coz}(\alpha) : \alpha \in R\}$. Also, for $B \subseteq Coz_c[L]$, we write $Coz_c^{\leftarrow}[B] = \{\alpha \in C_c(L) : \mathrm{coz}(\alpha) \in B\}$.
- We say that L is a countably P-frame (briefly, a CP-frame) if $C_c(L)$ is regular.
- $C^F(L) := \{\alpha \in \mathcal{R}(L) : R_\alpha \text{ is finite }\}$ is the pointfree topology version of the ring $C^F(X)$.
- $C^*_c(L) := \{\alpha \in \mathcal{R}(L) : \alpha \text{ is bounded and } R_\alpha \text{ is countable }\}$ is the pointfree topology version of the ring $C^*_c(X)$.
- For $\alpha \in C_c(L)$, we put $\mathrm{h}_{\mathrm{Max}}(\alpha) := \{M \in \max(C_c(L)) : \alpha \in M\}$.

Proposition 2.1. For a family $\{I_\lambda\}_{\lambda\in\Lambda}$ of z_c-ideals of $C_c(L)$, $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a z_c-ideal. Also, for an ideal I of $C_c(L)$, the following statements are equivalent.

(1) I is a z_c-ideal.
(2) For any $\alpha, \beta \in C_c(L)$, $\alpha \in I$ and $\mathrm{coz}(\beta) \leq \mathrm{coz}(\alpha)$ imply $\beta \in I$.
(3) $I = \bigcup\{\mathbf{M}^c_{\mathrm{coz}(\alpha)} : \alpha \in I\} = \sum_{\alpha\in I} \mathbf{M}^c_{\mathrm{coz}(\alpha)}$.

Proof . (1) $\Rightarrow$ (2). Suppose $\alpha \in I$ and that $\mathrm{coz}(\beta) \leq \mathrm{coz}(\alpha)$. In this case,

$$\mathrm{coz}(\beta) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \mathrm{coz}(\alpha\beta).$$

Since $\alpha\beta \in I$, it follows from the assumption that $\beta \in I$.

(2) $\Rightarrow$ (3). It is obvious that

$$I \subseteq \bigcup\{\mathbf{M}^c_{\mathrm{coz}(\alpha)} : \alpha \in I\}.$$

Now, suppose $\alpha \in I$ and $\beta \in \mathbf{M}^c_{\mathrm{coz}(\alpha)}$. Therefore, $\mathrm{coz}(\beta) \leq \mathrm{coz}(\alpha)$, and by the assumption, $\beta \in I$. Hence, $\mathbf{M}^c_{\mathrm{coz}(\alpha)} \subseteq I$, which establishes the reverse inclusion as well.

(3) $\Rightarrow$ (1). Suppose $\alpha \in I$ and $\beta \in C_c(L)$ are such that $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$. Then, $\beta \in \mathbf{M}^c_{\mathrm{coz}(\beta)} = \mathbf{M}^c_{\mathrm{coz}(\alpha)} \subseteq I$. □

Lemma 2.2. Suppose α is an invertible element in $\mathcal{R}(L)$. If $\alpha \in C_c(L)$, then $\alpha^{-1} \in C_c(L)$.

Proof . Suppose $0 \neq r \in \mathbb{R}$ is invertible. In this case,

$$\begin{aligned} r \in R_\alpha &\Leftrightarrow \operatorname{coz}(\alpha - \mathbf{r}) \neq \top \\ &\Leftrightarrow \operatorname{coz}(\alpha^{-1}) \wedge \operatorname{coz}(\alpha - \mathbf{r}) \neq \top, \qquad \text{since } \operatorname{coz}(\alpha^{-1}) = \top \\ &\Leftrightarrow \operatorname{coz}(\alpha^{-1})(\alpha - \mathbf{r}) \neq \top \\ &\Leftrightarrow \operatorname{coz}(-\mathbf{r})(\alpha^{-1} - \frac{\mathbf{1}}{\mathbf{r}}) \neq \top \\ &\Leftrightarrow \operatorname{coz}(-\mathbf{r}) \wedge \operatorname{coz}(\alpha^{-1} - \frac{\mathbf{1}}{\mathbf{r}}) \neq \top \\ &\Leftrightarrow \operatorname{coz}(\alpha^{-1} - \frac{\mathbf{1}}{\mathbf{r}}) \neq \top, \qquad \text{since } \operatorname{coz}(-\mathbf{r}) = \top \\ &\Leftrightarrow \frac{1}{r} \in R_{\alpha^{-1}}. \end{aligned}$$

□

Remark 2.3. We recall from [2] that a subring R of $\mathbb{R}^X$ is said to be *invertible* if, whenever $f \in R$ satisfies $Z(f) = \emptyset$, it follows that f is invertible in R.

As pointed out by the referee, upon introducing the following definition, Lemma 2.2 immediately implies that the subring $C_c(L)$ is inverse-closed in $\mathcal{R}(L)$.

Definition 2.4. Let S be a ring and let R be a subring of S. We say that R is an *invertible subring* (or *inverse-closed subring*) of S if

$$U(S) \cap R = U(R),$$

where $U(R)$ denotes the group of units of R. If the ambient ring S is clear from the context, we simply say that R is an *invertible subring* (or *inverse-closed subring*).

Lemma 2.5. Every maximal ideal of the ring $C_c(L)$ is a z_c-ideal.

Proof . Suppose I is a maximal ideal of the ring $C_c(L)$, and also suppose $\gamma \in C_c(L)$, $\beta \in I$ are such that $\operatorname{coz}(\beta) = \operatorname{coz}(\gamma)$, and $\gamma \notin I$. Since the ideal I is maximal, there exist $\alpha \in C_c(L)$ and $\psi \in I$ such that $\mathbf{1} = \psi + \alpha\gamma$. Therefore,

$$\top = \operatorname{coz}(\psi + \alpha\gamma) \leq \operatorname{coz}(\psi) \vee (\operatorname{coz}(\alpha) \wedge \operatorname{coz}(\gamma)) \leq \operatorname{coz}(\psi) \vee \operatorname{coz}(\gamma) = \operatorname{coz}(\psi) \vee \operatorname{coz}(\beta).$$

Hence, $\operatorname{coz}(\psi) \vee \operatorname{coz}(\beta) = \top$, and consequently, $\operatorname{coz}(\psi^2 + \beta^2) = \top$. Now, by Lemma 2.2, $\psi^2 + \beta^2$ is invertible in the ring $C_c(L)$, which contradicts the fact that $\psi^2 + \beta^2 \in I$. Therefore, $\gamma \in I$. □

Proposition 2.6. The following statements are true for any frame L.

(1) If I is a proper ideal of $C_c(L)$, then $\mathrm{Coz}_c[I]$ is a proper ideal of $\mathrm{Coz}_c[L]$.
(2) If I is a proper ideal of $Coz_c[L]$, then $\mathrm{Coz}_c^{\leftarrow}[I]$ is a proper ideal of $C_c(L)$.
(3) If M is a maximal ideal of $C_c(L)$, then $\mathrm{Coz}_c[M]$ is a maximal ideal of $\mathrm{Coz}_c[L]$.
(4) If M is a maximal ideal of $\mathrm{Coz}_c[L]$, then $\mathrm{Coz}_c^{\leftarrow}[M]$ is a maximal ideal of $C_c(L)$.

Proof. (1). Assume that the ideal I of the ring $C_c(L)$ is maximal, and that $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \in \mathrm{Coz}_c[I]$. In this case,

$$\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) = \mathrm{coz}(\alpha^2 + \beta^2) \in \mathrm{Coz}_c[I].$$

On the other hand, suppose $\mathrm{coz}(\alpha) \leq \mathrm{coz}(\beta)$ and $\mathrm{coz}(\beta) \in \mathrm{Coz}_c[I]$. Then,

$$\mathrm{coz}(\alpha) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \mathrm{coz}(\alpha\beta) \in \mathrm{Coz}_c[I].$$

Thus, $\mathrm{Coz}_c[I]$ is an ideal of $\mathrm{Coz}_c[L]$. If $\mathrm{Coz}_c[I]$ is not maximal, then there exists $\gamma \in I$ such that $\mathrm{coz}(\gamma) = \top$. Therefore, $\gamma \in I$ is invertible, which is a contradiction.

(2). Assume $\alpha, \beta \in \mathrm{Coz}_c^{\leftarrow}[I]$. Then $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \in I$ and also, $\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) \in I$. Hence,

$$\mathrm{coz}(\alpha + \beta) \leq \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) \in I,$$

so $\mathrm{coz}(\alpha + \beta) \in I$.

Now suppose $\alpha \in \mathrm{Coz}_c^{\leftarrow}[I]$ and $\gamma \in C_c(L)$. In this case, $\mathrm{coz}(\alpha) \in I$ and $\mathrm{coz}(\gamma) \in \mathrm{Coz}_c(L)$. Also,

$$\mathrm{coz}(\alpha) \geq \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\gamma) = \mathrm{coz}(\alpha\gamma).$$

Therefore, $\mathrm{coz}(\alpha\gamma) \in I$, which implies $\alpha\gamma \in \mathrm{Coz}_c^{\leftarrow}[I]$.

If $\mathrm{Coz}_c^{\leftarrow}[I]$ is not maximal, then there exists an invertible member $\beta \in C_c(L)$ such that $\beta \in \mathrm{Coz}_c^{\leftarrow}[I]$. Hence, $\top = \mathrm{coz}(\beta) \in I$, which is a contradiction.

(3). Suppose M is a maximal ideal of $C_c(L)$, and J is a proper ideal of $\mathrm{Coz}_c[L]$ such that $\mathrm{Coz}_c[M] \subseteq J$. Since M is a maximal ideal, by Lemma 2.5, $M = \mathrm{Coz}_c^{\leftarrow}[\mathrm{Coz}_c[M]]$, and also,

$$M = \mathrm{Coz}_c^{\leftarrow}[\mathrm{Coz}_c[M]] \subseteq \mathrm{Coz}_c^{\leftarrow}[J] \subseteq \mathrm{Coz}_c[L].$$

From the maximality of M, we conclude that $\mathrm{Coz}_c^{\leftarrow}[J] = M$. Hence, $\mathrm{Coz}_c[M] = J$.

(4). Assume $\alpha \notin \mathrm{Coz}_c^{\leftarrow}[M]$. Since $\mathrm{coz}(\alpha) \notin M$, there exists $b \in M$ such that $\mathrm{coz}(\alpha) \vee b = \top$. Because M is an ideal of $\mathrm{Coz}_c[L]$, there exists $\gamma \in C_c(L)$ such that $\mathrm{coz}(\gamma) = b$. Then,

$$\top = \mathrm{coz}(\alpha) \vee b = \mathrm{coz}(\alpha) \vee \mathrm{coz}(\gamma) = \mathrm{coz}(\alpha^2) \vee \mathrm{coz}(\gamma^2) = \mathrm{coz}(\alpha^2 + \gamma^2).$$

Consequently, by 2.2, $\alpha^2 + \gamma^2$ is invertible in $C_c(L)$. Therefore, for every $\alpha \in C_c(L) \setminus \mathrm{Coz}_c^{\leftarrow}[M]$, the ideal $\langle \alpha, \mathrm{Coz}_c^{\leftarrow}[M] \rangle$ in the ring $C_c(L)$ is not proper. Hence, $\mathrm{Coz}_c^{\leftarrow}[M]$ is a maximal ideal of the ring $C_c(L)$. □

Proposition 2.7. The following statements are true for any frame L.

(1) Let I be a proper ideal and P be a prime ideal in $C_c(L)$. If $I \cap P$ is a z_c-ideal and $I \nsubseteq P$, then P is a z_c-ideal.
(2) Let I be an ideal and P be a prime ideal in $C_c(L)$ such that $P \cap I$ is a z_c-ideal. Then, I or P is a z_c-ideal.
(3) Let P and Q be prime ideals in $C_c(L)$ that are not in a chain. If $P \cap Q$ is a z_c-ideal, then both P and Q are the z_c-ideal.

Proof . (1). Assume $\alpha \in P$, $\beta \in C_c(L)$, and also $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$. It is evident that there exists γ such that $\gamma \in I \setminus P$. However, $\mathrm{coz}(\alpha\gamma) = \mathrm{coz}(\beta\gamma)$ and $\alpha\gamma \in P \cap I$. Since $P \cap I$ is a z_c-ideal, we conclude that $\beta\gamma \in P \cap I$. Therefore, $\beta\gamma \in P$, which implies $\beta \in P$. Hence, P is a z_c-ideal.

(2). By the previous part, the proof is straightforward.

(3). Suppose $\alpha \in P$, $\beta \in C_c(L)$, and $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$. Since these two ideals do not form a chain, there exists γ such that $\gamma \in Q \setminus P$. Then, $\mathrm{coz}(\alpha\gamma) = \mathrm{coz}(\beta\gamma)$ and $\alpha\gamma \in P \cap Q$. From the fact that $P \cap Q$ is a z_c-ideal, we deduce that $\beta\gamma \in P \cap Q$. Thus, $\beta\gamma \in P$, which means $\beta \in P$. Therefore, P is a z_c-ideal. Similarly, it can be shown that Q is also a z_c-ideal.

□

Proposition 2.8. Suppose that R is an ℓ-ring (not necessarily a subring of $C(L)$). Then the following statements hold:

(a) For every $a, b \in R$, we have

$$a \leq a^+, \qquad -a \leq a^-, \qquad (a+b)^+ \leq a^+ + b^+, \qquad (a+b)^- \leq a^- + b^-.$$

(b) If, in addition, R is an ℓ-subring of $C(L)$, then for every $\alpha, \beta \in R$ we have

$$(\alpha + \beta)(0, +\infty) \leq \alpha(0, +\infty) \vee \beta(0, +\infty).$$

Proof . (a) The inequalities $a \leq a^+$ and $-a \leq a^-$ are clear. For the third inequality, we argue as follows:

$$(a+b)^+ = (a+b) \vee 0 \leq (a^+ + b) \vee a^+ = a^+ + (b \vee 0) = a^+ + b^+.$$

The proof of the fourth inequality is similar to that of the third one.

(b) By part (a), we may write

$$\begin{aligned}(\alpha+\beta)(0,+\infty) &= \operatorname{coz}\left((\alpha+\beta)^+\right)\\ &\leq \operatorname{coz}\left(\alpha^+ + \beta^+\right)\\ &= \operatorname{coz}(\alpha^+) \vee \operatorname{coz}(\beta^+)\\ &= \alpha(0,+\infty) \vee \beta(0,+\infty).\end{aligned}$$

□

Proposition 2.9. Let I be a proper z_c-ideal in $C_c(L)$. Then, the following statements are equivalent.

(1) I is a prime ideal in $C_c(L)$.
(2) I contains a prime ideal in $C_c(L)$.
(3) For all $\alpha, \beta \in C_c(L)$, if $\alpha\beta = \mathbf{0}$, then $\alpha \in I$ or $\beta \in I$.
(4) Given $\alpha \in C_c(L)$, there exists a cozero element $a \in \operatorname{Coz}_c[I]$ such that

$$\alpha(0,-) \leq a \quad \text{or} \quad \alpha(-,0) \leq a.$$

Proof . (1). The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Suppose $\alpha \in C_c(L)$. Since $\alpha^+\alpha^- = \mathbf{0}$ and $\alpha^+, \alpha^- \in C_c(L)$, we have $\alpha^+ \in I$ or $\alpha^- \in I$. Therefore, without loss of generality, suppose that $\alpha^+ \in I$. Set $a = \operatorname{coz}(\alpha^+)$. Then,

$$\alpha(0,-) \leq \alpha^+(0,-) = \operatorname{coz}(\alpha^+) = a.$$

(4) $\Rightarrow$ (1). Suppose $\alpha, \beta \in C_c(L)$ such that $\alpha\beta \in I$. Define $\gamma = |\alpha| - |\beta| \in C_c(L)$, and suppose $\delta \in I$ such that $\gamma(0,-) \leq \operatorname{coz}(\delta)$. Then, by Proposition 2.8, we have

$$\begin{aligned}\operatorname{coz}(|\alpha|) &= |\alpha|(0,-) = (\gamma + |\beta|)(0,-) \leq \gamma(0,-) \vee |\beta|(0,-) \leq |\delta|(0,-) \vee |\beta|(0,-)\\ &= \operatorname{coz}(|\delta| + |\beta|),\end{aligned}$$

and consequently,

$$\begin{aligned}\operatorname{coz}(\alpha) &= \operatorname{coz}(|\alpha|) = \operatorname{coz}(|\alpha|) \wedge \operatorname{coz}(|\delta| + |\beta|) = \operatorname{coz}(|\alpha\delta| + |\alpha\beta|)\\ &= \operatorname{coz}(\alpha\delta) \vee \operatorname{coz}(\alpha\beta),\end{aligned}$$

which belongs to $\mathrm{Coz}_c[I]$. Since I is a z_c-ideal, it follows that $\alpha \in I$.

Next, assume $\gamma(-,0) \leq \mathrm{coz}(\delta)$. Then, by Proposition 2.8, we have

$$\begin{aligned}\mathrm{coz}(|\beta|) &= |\beta|(0,-) = -|\beta|(-,0) = (\gamma - |\alpha|)(-,0) \leq \gamma(-,0) \vee (-|\alpha|(-,0)) \\ &= \gamma(-,0) \vee |\alpha|(0,-),\end{aligned}$$

and similarly,

$$\begin{aligned}\mathrm{coz}(\beta) &= \mathrm{coz}(|\beta|) = \mathrm{coz}(|\beta|) \wedge \mathrm{coz}(|\delta| + |\alpha|) = \mathrm{coz}(|\beta\delta| + |\beta\alpha|) \\ &= \mathrm{coz}(\beta\delta) \vee \mathrm{coz}(\beta\alpha),\end{aligned}$$

which belongs to $\mathrm{Coz}_c[I]$. By the z_c-ideal property of I, we conclude that $\beta \in I$. Thus, I is a prime ideal of the ring $C_c(L)$. □

Corollary 2.10. Let I be a proper ideal of $Coz_c[L]$ such that for every $\alpha, \beta \in C_c(L)$,

$$\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \perp \quad \text{implies} \quad \mathrm{coz}(\alpha) \in I \quad \text{or} \quad \mathrm{coz}(\beta) \in I.$$

Then, $Coz_c^{\leftarrow}[I]$ is a prime z_c-ideal of $C_c(L)$ if and only if I is a prime ideal of $Coz_c[L]$.

Proof . Suppose $\alpha, \beta \in C_c(L)$ and $\alpha\beta = \mathbf{0}$. Then, $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \perp$. By assumption, either $\mathrm{coz}(\alpha) \in I$ or $\mathrm{coz}(\beta) \in I$. Hence,

$$\alpha \in \mathrm{Coz}_c^{\leftarrow}[I] \quad \text{or} \quad \beta \in \mathrm{Coz}_c^{\leftarrow}[I].$$

Since $\mathrm{Coz}_c^{\leftarrow}[I]$ is a z_c-ideal of the ring $C_c(L)$, by Proposition 2.9, $\mathrm{Coz}_c^{\leftarrow}[I]$ is a prime z_c-ideal of the ring $C_c(L)$.

Conversely, suppose $\alpha, \beta \in C_c(L)$ and $\mathrm{coz}(\alpha\beta) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) \in I$. Then, $\alpha\beta \in \mathrm{Coz}_c^{\leftarrow}[I]$, and by assumption,

$$\alpha \in \mathrm{Coz}_c^{\leftarrow}[I] \quad \text{or} \quad \beta \in \mathrm{Coz}_c^{\leftarrow}[I].$$

Hence, $\mathrm{coz}(\alpha) \in I$ or $\mathrm{coz}(\beta) \in I$. Thus, I is a prime z_c-ideal of $\mathrm{Coz}_c[L]$. □

Proposition 2.11. Every prime ideal of $C_c(L)$ is included in a unique maximal ideal.

Proof . Every prime ideal P is contained in at least one maximal ideal. Now, suppose M and M' are two distinct maximal ideals such that $P \subseteq M$ and $P \subseteq M'$. Then, by Lemma 2.5 and Proposition 2.1, the intersection $M \cap M'$ is a z_c-ideal. Since $P \subseteq M \cap M'$, by

Proposition 2.9, we deduce that $M \cap M'$ is a prime ideal of the ring $C_c(L)$ and so either $M \subseteq M'$ or $M' \subseteq M$. Therefore, $M = M'$. □

According to [4, 15], a ring A is called a **Gelfand ring** if and only if, for every $a, b \in A$ such that $a + b = 1$, there exist $r, s \in A$ such that

$$(1 + ar)(1 + bs) = 0.$$

Moreover, a commutative ring R is Gelfand if and only if every prime ideal of R is contained in a unique maximal ideal.

As an application of Proposition 2.11, we state the following result.

Corollary 2.12. $C_c(L)$ is a Gelfand ring.

Let us conclude this section with the interesting observations that follow. Similar to z_c-ideals in $C_c(L)$, one can introduce z_F-ideals in $C^F(L) := \{\alpha \in \mathcal{R}(L) \colon R_\alpha \text{ is finite } \}$ as follows. An ideal I in $C^F(L)$ is called a z_F-ideal if, for every $\alpha \in C^F(L)$ and $\beta \in I$, $\operatorname{coz}(\alpha) = \operatorname{coz}(\beta)$ implies $\alpha \in I$. If we replace z_c-ideals with z_F-ideals, all the results obtained in this section remain valid for $C^F(L)$.

As noted earlier, the concept of z_R associated with an arbitrary subring R of $\mathbb{R}^X$ was introduced in [2]. To simplify the notation, we shall consistently employ the above symbols in what follows.

3 On the regularity of $C_c(L)$ and CP-frames

A commutative ring A is called *regular* if for every $a \in A$, there exists some element $b \in A$ such that $a^2b = a$ and also, a is a *regular element* of a commutative ring A if and only if there exists an idempotent element $e \in R$ such that $a = ae$.

In this section, we introduce CP-frames, study the way they are related to P-frames, and provide alternative characterizations of CP-frames. Also, we obtain some results on CP-frames. Recall from [11, 4J] that X is a P-space if and only if $C(X)$ is a regular ring. Also, recall from the definition of countably P-spaces, presented in [10], that a topological space X is said to be a CP-space if $C_c(X)$ is regular. We say that L is a countably P-frame (briefly, a CP-frame) if $C_c(L)$ is regular (see [8]).

Remark 3.1. We recall from [6] that if the element a in the frame L is complemented and $\alpha \in \mathcal{R}(L)$, then the function $\beta : \mathcal{L}(\mathbb{R}) \to L$, defined by the rule

$$\beta(u) = \begin{cases} \alpha(u) \vee a' & \text{if } 0 \in \tau(u) \\ \\ \alpha(u) \wedge a & \text{if } 0 \notin \tau(u) \end{cases}$$

is a real-valued continuous function. Furthermore, according to [6, Proposition 4.2], we have $\beta = \alpha e_a$, where e_a is the continuous real-valued function introduced in [6, Proposition 4.1], that is, $e_a : \mathcal{L}(\mathbb{R}) \to L$ with the rule

$$e_a(u) = \begin{cases} \top & \text{if } 0, 1 \in \tau(u) \\ a' & \text{if } 0 \in \tau(u),\ 1 \notin \tau(u) \\ a & \text{if } 0 \notin \tau(u),\ 1 \in \tau(u) \\ \bot & \text{if } 0 \notin \tau(u),\ 1 \notin \tau(u) \end{cases}$$

It is clear that e_a is an idempotent element in $\mathcal{R}(L)$. Therefore, if $\alpha \in \mathcal{R}(L)$ and $\mathrm{coz}(\alpha)$ is complemented, then $\alpha = \alpha e_{\mathrm{coz}(\alpha)}$.

Proposition 3.2. Let L be a P-frame, and fix $\alpha \in C_c(L)$. we define $\iota_\alpha : \mathcal{L}(\mathbb{R}) \to L$ by

$$\iota_\alpha(v) = \begin{cases} \alpha(v^{-1}) & 0 \notin \tau(v) \\ \alpha(v_0^{-1}) \vee (\mathrm{coz}(\alpha))' & 0 \in \tau(v). \end{cases}$$

Then $\iota_\alpha \in C_c(L)$, where for every $v \in \mathcal{L}(\mathbb{R})$, if $0 \notin \tau(v)$, then

$$v^{-1} = \tau^{-1}(\{\frac{1}{x} : x \in \tau(v)\})$$

and if $0 \in \tau(v)$, then

$$v_0^{-1} = \tau^{-1}(\{\frac{1}{x} : x \in \tau(v), x \neq 0\}).$$

Proof . It is evident that $\iota_\alpha \in \mathcal{R}(L)$. Since $\alpha \in C_c(L)$, we conclude that R_α is a countable subset of $\mathbb{R}$. We claim that for every nonzero $r \in \mathbb{R}$,

$$r \in R_\alpha \iff \frac{1}{r} \in R_{\iota_\alpha}.$$

Let $r \in R_\alpha$ with $r > 0$. Suppose $u \in \mathcal{L}\mathbb{R}$ is such that $0 \in \tau(u)$. Then $\mathrm{coz}(\alpha) \vee \alpha(u) = \top$, which implies

$$\mathrm{coz}(\alpha)' \leq \alpha(u).$$

In particular, for every $n \in \mathbb{N}$,

$$\mathrm{coz}(\alpha)' \leq \alpha\left(-\tfrac{1}{n}, \tfrac{1}{n}\right).$$

Since this holds for all n, we may choose n such that $\frac{1}{n} < r$. Then

$$\begin{aligned}
\operatorname{coz}\left(\iota_\alpha - \tfrac{1}{r}\right) &= \iota_\alpha\left(-, \tfrac{1}{r}\right) \vee \iota_\alpha\left(\tfrac{1}{r}, -\right) \\
&= \alpha(-, 0) \vee \alpha(r, -) \vee (\operatorname{coz}(\alpha))' \vee \alpha(0, r) \\
&\leq \alpha(-, 0) \vee \alpha(r, -) \vee \alpha\left(-\tfrac{1}{n}, \tfrac{1}{n}\right) \vee \alpha(0, r) \\
&= \left[\alpha(-, 0) \vee \alpha\left(-\tfrac{1}{n}, \tfrac{1}{n}\right) \vee \alpha(r, -)\right] \vee \alpha(0, r) \\
&= \alpha(r, 0) \vee \alpha(0, r) \\
&= \operatorname{coz}(\alpha - r) \\
&\neq \top.
\end{aligned}$$

Thus $\frac{1}{r} \in R_{\iota_\alpha}$. Additionally, if r is negative, it can be easily shown that $\operatorname{coz}(\iota_\alpha - \frac{1}{r}) \neq \top$. Now, assume that $\frac{1}{r} \in R_{\iota_\alpha}$ and also, r is positive.

$$\begin{aligned}
\frac{1}{r} \in R_{\iota_\alpha} &\Rightarrow \operatorname{coz}(\iota_\alpha - \frac{1}{r}) \neq \top \\
&\Rightarrow \iota_\alpha(-, \frac{1}{r}) \vee \iota_\alpha(\frac{1}{r}, -) \neq \top \\
&\Rightarrow \alpha(-, 0) \vee \alpha(r, -) \vee (\operatorname{coz}(\alpha))' \vee \alpha(0, r) \neq \top \\
&\Rightarrow \alpha(-, 0) \vee \alpha(r, -) \vee \alpha(0, r) \neq \top \\
&\Rightarrow \alpha(-, r) \vee \alpha(r, -) \neq \top \\
&\Rightarrow \operatorname{coz}(\alpha - r) \neq \top \\
&\Rightarrow r \in R_\alpha.
\end{aligned}$$

□

Similar to what has been established in the ring $\mathcal{R}(L)$, it can be observed that for every $\alpha \in C_c(L)$, if $\operatorname{coz}(\alpha)$ is complemented, then

$$\iota_\alpha \alpha = e_{\operatorname{coz}(\alpha)}.$$

Now, based on this fact and the previous proposition, we proceed to prove the next proposition.

Proposition 3.3. Every P-frame is a CP-frame.

Proof . Let L be a P-frame. We show that $C_c(L)$ is a regular ring. We assume that $\alpha \in C_c(L)$ and also, ι_α is a continuous real-valued function of the proposition 3.2. We show that $\alpha = \alpha^2 \iota_\alpha$. Let $\nu \in \mathcal{L}(\mathbb{R})$ be given. If $0 \in \tau(\nu)$, then

$$\alpha(\nu) \vee \operatorname{coz}(\alpha) = \top \;\Rightarrow\; (\operatorname{coz}(\alpha))' = \alpha(\nu) \wedge (\operatorname{coz}(\alpha))'.$$

Since L is a P-frame, $\mathrm{coz}(\alpha) \vee (\mathrm{coz}(\alpha))' = \top$. Hence,

$$\begin{aligned}
\alpha(v) &= \alpha(v) \wedge (\mathrm{coz}(\alpha) \vee (\mathrm{coz}(\alpha))') \\
&= (\alpha(v) \wedge \mathrm{coz}(\alpha)) \vee (\alpha(v) \wedge (\mathrm{coz}(\alpha))') \\
&= \alpha(\tau^{-1}(\tau(v) \setminus \{0\})) \vee (\mathrm{coz}(\alpha))' \\
&= \alpha(v) \vee (\mathrm{coz}(\alpha))'.
\end{aligned}$$

Therefore, according to Proposition 3.1 and the previous discussion, for each $v \in \mathcal{L}(\mathbb{R})$, we have:

$$\begin{aligned}
\alpha^2 \iota_\alpha(v) &= \alpha e_{\mathrm{coz}(\alpha)}(v) \\
&= \begin{cases} \alpha(v) \vee (\mathrm{coz}(\alpha))' & 0 \in \tau(v) \\ \alpha(v) \wedge \mathrm{coz}(\alpha) & 0 \notin \tau(v) \end{cases} \\
&= \begin{cases} \alpha(v) & 0 \in \tau(v) \\ \alpha(v) & 0 \notin \tau(v) \end{cases} \\
&= \alpha(v).
\end{aligned}$$

Thus, the ring $C_c(L)$ is regular. Consequently, L is a CP-frame. □

Example 3.4. Let $X = [-1, 0] \bigcup \mathbb{N}$ be a subspace of the set $\mathbb{R}$ of the real numbers. Then, X is a CP-space which is not a P-space.

Proposition 3.5. In any space X, $C_c(\mathfrak{O}X) \cong C_c(X)$.

Proof . We recall from [1] that

$$\mathcal{O} : C_c(X) \to C_c(\mathcal{O}X), \quad \forall U \in \mathcal{O}X,\ \forall f \in C_c(X), \quad (\mathcal{O}f)(U) = f^{-1}(U)$$

is an algebraic and lattice isomorphism. Since $C_c(X)$ is a sublattice of $C(X)$, it is sufficient here to show that $R_f = R_{\mathcal{O}f}$. For this purpose, we can write:

$$r \in R_f \iff r \in f(X) \iff f^{-1}(R_r) \neq X \iff (\mathcal{O}f)(R_r) \neq \top \iff r \in R_{\mathcal{O}f}.$$

□

Lemma 3.6. Assume that L is a frame and A is a regular subring of the ring $\mathcal{R}(L)$. Then, for every $\alpha \in A$, the element $\mathrm{coz}(\alpha)$ is complemented in $\mathrm{Coz}[L]$.

Proof . Suppose $\alpha \in A$. Since A is a regular ring, there exists $\beta \in A$ such that $\alpha = \alpha^2\beta$. Therefore,

$$\mathrm{coz}(\alpha) = \mathrm{coz}(\alpha^2\beta) = \mathrm{coz}(\alpha^2) \wedge \mathrm{coz}(\beta) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \mathrm{coz}(\alpha\beta).$$

Set $e = \alpha\beta$. It is clear that e is an idempotent element of the ring $\mathcal{R}(L)$. Hence, we have

$$\mathrm{coz}(e) \wedge \mathrm{coz}(e - \mathbf{1}) = \mathrm{coz}(e(e - \mathbf{1})) = \mathrm{coz}(\mathbf{0}) = \bot,$$

and

$$\mathrm{coz}(e) \vee \mathrm{coz}(e - \mathbf{1}) = \mathrm{coz}(e^2 + (e - \mathbf{1})^2) = \mathrm{coz}(\mathbf{1}) = \top.$$

Thus, $\mathrm{coz}(\alpha)$ is complemented in $\mathrm{Coz}[L]$. □

Clearly, a topological space X is a P-space if and only if the frame $\mathfrak{O}X$ is a P-frame. By Lemma 3.5, we showed that for any space X, $C_c(X) \cong C_c(\mathfrak{O}X)$. This means that $C_c(\mathfrak{O}X)$ is isomorphic to a $C_c(X)$ via an f-algebra isomorphism. Hence, a topological space X is a CP-space if and only if the frame $\mathfrak{O}X$ is a CP-frame. Thus, by this fact and by Proposition 3.3, if X is a P-space, then the frame $\mathfrak{O}X$ is a CP-frame. Therefore, in the above example, $\mathfrak{O}X$ is a CP-frame which is not a P-frame.

Proposition 3.7. A frame L is a CP-frame if and only if every $\mathrm{coz}(\alpha) \in Coz_c[L]$ is complemented in $Coz_c[L]$.

Proof . Necessity. Suppose that L is a CP-frame and $\alpha \in C_c(L)$. Since the ring $C_c(L)$ is regular, there exists $\beta \in C_c(L)$ such that $\alpha = \alpha^2\beta$. Hence, we have $\mathrm{coz}(\alpha) = \mathrm{coz}(\alpha\beta)$, and $\mathrm{coz}(\alpha\beta) \wedge \mathrm{coz}(\alpha\beta - \mathbf{1}) = \bot$, and $\mathrm{coz}(\alpha\beta) \vee \mathrm{coz}(\alpha\beta - \mathbf{1}) = \top$. Therefore, $\mathrm{coz}(\alpha)$ is a complemented element in $\mathrm{Coz}_c[L]$. Hence, L is a P-frame.

Sufficiency. Assume that $\mathrm{coz}(\alpha)$ is complemented for every $\alpha \in C_c(L)$. We aim to show that the ring $C_c(L)$ is regular. Since $\mathrm{coz}(\alpha)$ is complemented, then $\mathrm{coz}(\alpha) \vee (\mathrm{coz}(\alpha))^* = \top$ and $\mathrm{coz}(\alpha) \wedge (\mathrm{coz}(\alpha))^* = \bot$. Now, consider the map $\iota_\alpha : \mathcal{L}(\mathbb{R}) \to L$ from the Proposition 3.2. By a similar argument as in the proof of Proposition 3.3, we have $\alpha^2\iota_\alpha = \alpha$, and therefore, the ring $C_c(L)$ is regular. Consequently, L is a CP-frame. □

Now, we state a proposition regarding regular rings, which we will need: items (2) and (3) are from the source [12], and (4), (5), and (6) are from [16].

Proposition 3.8. For a reduce ring A, the following statements are equivalent:

(1) The ring A is regular.
(2) Every principal ideal is generated by an idempotent element.
(3) Every prime ideal is maximal.
(4) Every ideal is an intersection of maximal ideals.
(5) Every ideal is a z-ideal in Mason's sense.
(6) Every principal ideal is a z-ideal in Mason's sense.

Theorem 3.9. The following statements are equivalent for a frame L.

(1) The frame L is a CP-frame.
(2) Every ideal of $C_c(L)$ is a z-ideal in Mason's sense..
(3) Every ideal of $C_c(L)$ is a z_c-ideal.
(4) Every ideal of $C_c(L)$ is an intersection of prime ideals.
(5) Every ideal of $C_c(L)$ is an intersection of maximal ideals.
(6) Every prime ideal of $C_c(L)$ is an intersection of maximal ideals.
(7) For every $\alpha, \beta \in C_c(L)$, $< \alpha, \beta >=< \alpha^2 + \beta^2 >$.
(8) Every principal ideal of $C_c(L)$ is generated by an idempotent.
(9) Every prime ideal of $C_c(L)$ is maximal.
(10) For every $\alpha \in C_c(L)$, $\mathrm{coz}(\alpha) \vee (\mathrm{coz}(\alpha))^* = \top$.

Proof . Since the ring $C_c(L)$ is reduce, the truth of statements (1), (2), (5), (8), and (9) follows immediately from Proposition 3.8.

(2) $\Rightarrow$ (3). Suppose that I is an ideal of the ring $C_c(L)$. Let $(\alpha, \beta) \in I \times C_c(L)$ with $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ be given. It is evident that $Q \in \mathrm{h_{Max}}(\alpha)$ if and only if Q is a maximal ideal of the ring $C_c(L)$ and $\alpha \in Q$ if and only if Q is a maximal ideal of the ring $C_c(L)$ and $\beta \in Q$ if and only if $Q \in \mathrm{h_{Max}}(\beta)$. Hence, $\mathrm{h_{Max}}(\alpha) = \mathrm{h_{Max}}(\beta)$, which implies from definition of a z-ideal that $\beta \in I$. Therefore, I is a z_c-ideal.

(3) $\Rightarrow$ (1). Let $\alpha \in C_c(L)$ be given. Since $\mathrm{coz}(\alpha^2) = \mathrm{coz}(\alpha)$, every ideal contains α if and only if it contains α^2. Then $< \alpha >=< \alpha^2 >$, which implies that there exists an element $\beta \in C_c(L)$ such that $\alpha = \alpha^2\beta$. Therfore, L is a CP-frame.

(1) $\Rightarrow$ (4). By Proposition 3.8, it is evident.

(4) $\Rightarrow$ (1). Every prime ideal of the ring $C_c(L)$ contains φ if and only if it contains φ^2. Thus, by statement (3), we have $< \varphi >=< \varphi^2 >$. Hence, for some $\alpha \in C_c(L)$, it holds that $\varphi = \alpha\varphi^2$. As a result, the ring $C_c(L)$ is von Neumann regular, and consequently, L is a CP-frame.

(5) $\Rightarrow$ (6). This is obvious.

(6) $\Rightarrow$ (1). By Proposition 2.11, each prime ideal of the ring $C_c(L)$ is contained within a unique maximal ideal. Therefore, under our assumptions, every prime ideal of $C_c(L)$ is maximal. Hence, by Proposition 3.8, the ring $C_c(L)$ is von Neumann regular, i.e., L is a CP-frame.

(3) $\Rightarrow$ (7). Suppose that $\alpha, \beta \in C_c(L)$. Then, $< \alpha^2 + \beta^2 > \subseteq < \alpha, \beta > \in \mathrm{Id}(C_c(L))$. Let $\varphi \in < \alpha, \beta >$. Then, there exist $\gamma_1, \gamma_2 \in C_c(L)$ such that $\varphi = \gamma_1\alpha + \gamma_2\beta$. Thus,

$$\mathrm{coz}(\varphi) = \mathrm{coz}(\gamma_1\alpha + \gamma_2\beta) \leq \mathrm{coz}(\gamma_1\alpha) \vee \mathrm{coz}(\gamma_2\beta) \leq \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) = \mathrm{coz}(\alpha^2 + \beta^2).$$

By statement (3), $< \alpha^2 + \beta^2 >$ is a z_c-ideal, so $\varphi \in < \alpha^2 + \beta^2 >$. Therefore, $< \alpha, \beta > \subseteq < \alpha^2 + \beta^2 >$, and so, $< \alpha, \beta > = < \alpha^2 + \beta^2 >$.

(7) $\Rightarrow$ (1). Suppose $\alpha \in C_c(L)$. Then,

$$< \alpha > = < \alpha, \mathbf{0} > = < \alpha^2 > .$$

Therefore, $\alpha \in < \alpha^2 > \in \mathrm{Id}(C_c(L))$. Hence, there exists $\beta \in C_c(L)$ such that $\alpha = \beta\alpha^2$. Thus, the ring $C_c(L)$ is von Neumann regular, which implies that L is a CP-frame.

(10) $\Leftrightarrow$ (1). By the equivalence of statements (1) and (2), and Proposition 3.7, this is obvious. □

Corollary 3.10. Let L be a CP-frame. Then, every finitely generated ideal of $C_c(L)$ is generated by an idempotent.

For a subset A of a frame L, $\mathcal{P} := \{\uparrow a : \ a \in A\}$ is called an ideal of closed quotients of L if it satisfies the following two properties.

- $\uparrow x \subseteq \uparrow y$ and $\uparrow y \in \mathcal{P}$ imply $\uparrow x \in \mathcal{P}$.
- If $\uparrow x, \uparrow y \in \mathcal{P}$, then $\uparrow(x \wedge y) \in \mathcal{P}$.

For every ideal $\mathcal{P}$ of closed quotients of L, we set

$$C_{c\mathcal{P}}(L) = \{\alpha \in \mathcal{R}_c(L) : \ \uparrow(\mathrm{coz}(\alpha))^* \in \mathcal{P}\}.$$

Lemma 3.11. For every ideal $\mathcal{P}$ of closed quotients of L, the following statements are true.

(1) $C_{c\mathcal{P}}(L)$ is a z_c-ideal of $C_c(L)$.
(2) $C_{c\mathcal{P}}(L)$ is proper if and only if $L \notin \mathcal{P}$.

Proof . (1). Suppose $\alpha, \beta \in C_{c\mathcal{P}}(L)$. Therefore,

$$\operatorname{coz}(\alpha - \beta) \leq \operatorname{coz}(\alpha) \vee \operatorname{coz}(\beta).$$

Additionally,

$$(\operatorname{coz}(\alpha))^* \wedge (\operatorname{coz}(\beta))^* \leq (\operatorname{coz}(\alpha - \beta))^*.$$

Since $\mathcal{P}$ is an ideal of the complement of closed subsets of the lattice L, it follows that

$$\uparrow(\operatorname{coz}(\alpha - \beta))^* \leq \uparrow((\operatorname{coz}(\alpha))^* \wedge (\operatorname{coz}(\beta))*) \in \mathcal{P}.$$

Hence, $\uparrow(\operatorname{coz}(\alpha - \beta))^* \in \mathcal{P}$, and therefore, $\alpha - \beta \in C_{c\mathcal{P}}(L)$. Similarly, it can be shown that $\alpha\beta \in C_{c\mathcal{P}}(L)$. Thus, $C_{c\mathcal{P}}(L)$ is an ideal of the ring $C_c(L)$.

Next, we show that $C_{c\mathcal{P}}(L)$ is a z_c-ideal. Suppose $\operatorname{coz}(\beta) = \operatorname{coz}(\alpha)$, with $\alpha \in C_{c\mathcal{P}}(L)$ and $\beta \in C_c(L)$. Then $\uparrow((\operatorname{coz}(\alpha))^*) \in \mathcal{P}$, it follows that $\uparrow(\operatorname{coz}(\beta))^* \in \mathcal{P}$. Consequently, $\beta \in C_{c\mathcal{P}}(L)$. Therefore, $C_{c\mathcal{P}}(L)$ is a z_c-ideal.

(2). Since $\uparrow(\operatorname{coz}(\mathbf{1}))^* = L$, it follows that $L \notin \mathcal{P}$ if and only if $\mathbf{1} \notin C_{c\mathcal{P}}(L)$. □

The next result describes a CP-frame L via its ideals of closed quotients.

Proposition 3.12. Let L be a frame. Then, L is a CP-frame if and only if for every ideal I of $C_c(L)$, there exists an ideal $\mathcal{P}$ of closed quotients of L such that $I = C_{c\mathcal{P}}(L)$.

Proof . Necessity. Let L be a CP-frame, and I be an ideal of the ring $C_c(L)$. Define

$$\mathcal{P} := \{\uparrow a \colon a \in L \text{ and there exists } \alpha \in I \text{ such that } \uparrow a \leq \uparrow(\operatorname{coz}(\alpha))^*\}.$$

Let $x, y \in L$ be given.

- If $\uparrow x \subseteq \uparrow y$ and $\uparrow y \in \mathcal{P}$, then there exists an element α in I such that $\uparrow y \leq \uparrow(\operatorname{coz}(\alpha))^*$, it follows that $\uparrow y \leq \uparrow(\operatorname{coz}(\alpha))^*$, and so $\uparrow x \in \mathcal{P}$.
- If $\uparrow x, \uparrow y \in \mathcal{P}$, then there exist $\alpha, \beta \in I$ such that $\uparrow x \leq \uparrow(\operatorname{coz}(\alpha))^*$ and $\uparrow y \leq \uparrow(\operatorname{coz}(\beta))^*$. Since $\alpha^2 + \beta^2 \in I$ and

$$\begin{aligned}
\uparrow(x \wedge y) &= \uparrow x \vee \uparrow y \leq \uparrow(\operatorname{coz}(\alpha))^* \vee \uparrow(\operatorname{coz}(\beta))^* \\
&= \uparrow\left((\operatorname{coz}(\alpha))^* \wedge (\operatorname{coz}(\beta))^*\right) \\
&= \uparrow\left(\operatorname{coz}(\alpha) \vee \operatorname{coz}(\beta)\right)^* \\
&= \uparrow\left(\operatorname{coz}(\alpha^2 + \beta^2)\right)^*,
\end{aligned}$$

we conclude that $\uparrow(x \wedge y) \in \mathcal{P}$.

Hence, $\mathcal{P}$ is an ideal of closed quotients of L. We claim that $I = C_{c\mathcal{P}}(L)$. Obviously, $I \subseteq C_{c\mathcal{P}}(L)$. Now, assume $\alpha \in C_{c\mathcal{P}}(L)$. Therefore, $\uparrow(\mathrm{coz}(\alpha))^* \in \mathcal{P}$, so there exists $\beta \in I$ such that $\uparrow(\mathrm{coz}(\alpha))^* \leq \uparrow(\mathrm{coz}(\beta))^*$. Hence, $(\mathrm{coz}(\beta))^* \leq (\mathrm{coz}(\alpha))^*$, and according to Theorem 3.9,

$$\mathrm{coz}(\alpha) = (\mathrm{coz}(\alpha))^{**} \leq (\mathrm{coz}(\beta))^{**} = \mathrm{coz}(\beta).$$

Since I is a z_c-ideal, it follows that $\alpha \in I$.

Sufficiency. Assume that I is an ideal of the ring $C_c(L)$. Then, by hypothesis, there exists an ideal $\mathcal{P}$ of closed quotients of L such that $I = C_{c\mathcal{P}}(L)$. Thus, by Lemma 3.11, I is a z_c-ideal, and by Theorem 3.9, L is a CP-frame. □

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Conflict of interest

The authors declare that they have no conflict of interest.

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