Journal of Frame and Matrix Theory (JFMT) Vol. 3 (2025) No. 1, 20–36
Online ISSN: 3041-8526
Faculty of Mathematics and Computer Science
Hakim Sabzevari University
http://dx.doi.org/10.22034/jfmt.2026.558228.1031

© **Research paper**

Some probability inequalities in Riesz spaces

Mahin Sadat Divandar [a,*], Ghadir Sadeghi [b]

[a] *Farhangian University of Sabzevar, Razavi Khorasan Province, Sabzevar, Iran*

[b] *Department of Mathematics and Computer Science, Hakim Sabzevari University, Sabzevar, Iran*

(Communicated by Reza Saadati)

Article Info

Received: 8 November 2025
Revised: 17 August 2026
Accepted: 24 August 2026

Keywords:
Riesz space,
Probability inequalities,
Conditional expectation,
Weak order unit.

2020 MSC:
46A40, 46B40, 60E15, 60B11.

Abstract

In this paper, we extend several classical probability inequalities to the Riesz space setting. We provide Marshalls inequality in Riesz spaces. Marshalls inequality is stronger than Kolmogorovs inequality. It gives a bound on the probability for the partial sums of a sequence of independent random variables. Also, we state an inequality of maximal increment in Riesz spaces, which is known as Shao's inequality. In addition, we find a better estimate for Etemadis inequality in vector lattices. Furthermore, we establish weak symmetrization inequalities and strong symmetrization inequalities in a measure-free framework. In the weak symmetrization inequalities, we state the relation between tail probabilities of random variables and tail probabilities of their symmetrizations. In the strong symmetrization inequalities, we extend the weak symmetrization inequalities to maximal inequalities.

*Corresponding author
Email addresses: m.divandar@hsu.ac.ir; md.divandar@gmail.com (Mahin Sadat Divandar), g.sadeghi@hsu.ac.ir; ghadir54@gmail.com (Ghadir Sadeghi)

©Authors retain the copyright and full publishing rights.

1 Introduction

The study of stochastic processes and probability theory is one of the most interesting applications in the theory of vector lattices (Riesz spaces). There is a correspondence between arguments in the classical setting of probability theory and the probability theory of Riesz spaces. In fact, measure-theoretic concepts can be replaced by order-theoretic concepts. Events are traditionally defined as measurable sets and they are associated with band projections. Hence, the σ-algebra of measurable sets is associated with a Boolean algebra of band projections. Furthermore, random variables are traditionally defined in terms of measurable functions; hence, it is a natural assumption to consider them as members of a Riesz space. The general theory of stochastic processes in Riesz spaces has been considered by [8, 9, 10, 13, 15, 16, 21, 22, 23]. It has also been studied in [2, 3, 14, 7]. In the probability theory of Riesz spaces, the role of the probability measure is played by a conditional expectation operator, which was defined by Kuo, Labuschagne, and Watson as a strictly positive order-continuous projection mapping weak order units to weak order units and having a Dedekind complete range. A more detailed explanation is given in [15, 16]. The concept of conditional independence was generalized from the classical setting of probability theory to the framework of Riesz spaces in [17]. This issue has also been explored by other authors in [18, 19]. Some important inequalities, such as Chebyshev's inequality, Jensen's inequality, Doob's martingale inequalities, Hájek-Rényi-Chow maximal inequality, Kolmogorov's inequality, Lévy's inequality, Etemadi's inequality, etc., have been studied in the probability theory of Riesz spaces setting (see [6, 11, 12, 20]). Probability inequalities play an important role in probability theory, since estimating the probabilities of an event or the sum of random variables helps us to understand the behavior of the elements. In this paper, we extend several classical probability inequalities to the Riesz space setting. We provide Marshalls inequality in Riesz spaces, which is stronger than Kolmogorovs inequality. It gives a bound on the probability for the partial sums of a sequence of independent random variables. Also, we state an inequality of maximal increment in Riesz spaces, which is known as Shao's inequality. In addition, we find a better estimate for Etemadis inequality in vector lattices. Furthermore, we establish weak symmetrization inequalities and strong symmetrization inequalities in a measure-free framework. In the weak symmetrization inequalities, we state the relation between tail probabilities of random variables and tail probabilities of their symmetrizations. In the strong symmetrization inequalities, we extend the weak symmetrization inequalities to maximal inequalities.

2 Preliminaries

We assume the reader to be familiar with the basic notations of Riesz spaces and we refer the reader to [1, 25]. For the convenience of the reader, we recall some of the main

concepts.
Let $\mathscr{E}$ be a Riesz space and $E \in \mathscr{E}$, $E > 0$. We say that E is a weak order unit for $\mathscr{E}$ if the band generated by E is $\mathscr{E}$. The Riesz space $\mathscr{E}$ is called a Dedekind complete Riesz space whenever every nonempty subset that is order bounded above has a supremum.
Let $\mathscr{E}$ be a Riesz space with a weak order unit E. A strictly positive order-continuous projection $\mathbb{F}$ on $\mathscr{E}$ mapping weak order units to weak order units and having a Dedekind complete range is called a *conditional expectation operator* on $\mathscr{E}$. The Riesz space $\mathscr{E}$ is called $\mathbb{F}$-*universally complete* if for each increasing net (X_α) in $\mathscr{E}_+$ with $(\mathbb{F}X_\alpha)$ order bounded in $\mathscr{E}^u$, we have that (X_α) is order convergent in $\mathscr{E}$ to X. It is known that $\mathscr{E}_E$, the principal ideal generated by E, is lattice isomorphic to a space $C(K)$ with K a compact Hausdorff space, such that E corresponds to the constant random variable 1 (see [1]). Note that $\mathscr{E}_E$ can be endowed with an f-algebra structure which coincides with the f-algebra structure of $C(K)$. This multiplication can be uniquely extended to $\mathscr{E}^u$, in which E is both a multiplicative unit and a weak order unit. This multiplication is constructed by setting

$$(\mathbb{P}E).(\mathbb{Q}E) = \mathbb{P}\mathbb{Q}E = (\mathbb{Q}E).(\mathbb{P}E),$$

for two band projections $\mathbb{P}$ and $\mathbb{Q}$. For more details about f-algebras, one can see [25, 4, 5].
Let X and Y be two elements of $\mathscr{E}$. The band generated by $(Y-X)^+$ is denoted by $\mathcal{B}(X < Y)$ and $\mathbb{P}_{(Y-X)^+}$ is called its corresponding band projection. Furthermore, $\mathcal{B}(Y \leq X)$, with corresponding band projection $\mathbb{P}_{(Y-X)^-}$, is the disjoint complement of the band $\mathcal{B}(X < Y)$.
Let $\mathbb{F}$ be a conditional expectation on $\mathscr{E}$ with the range $\mathfrak{F}$. Two band projections $\mathbb{P}$ and $\mathbb{Q}$ in the Boolean algebra $\mathfrak{B}$ of all band projections of $\mathscr{E}$ are called $\mathbb{F}$-*conditionally independent* whenever

$$\mathbb{F}\mathbb{P}\mathbb{F}\mathbb{Q}E = \mathbb{F}\mathbb{P}\mathbb{Q}E = \mathbb{F}\mathbb{Q}\mathbb{F}\mathbb{P}E. \tag{2.1}$$

By [12], in (2.1) the second equality follows from the first equality. Equivalently, $\mathbb{P}$ and $\mathbb{Q}$ are $\mathbb{F}$-conditionally independent whenever

$$\mathbb{F}\mathbb{P}\mathbb{F}\mathbb{Q}|_{\mathfrak{F}} = \mathbb{F}\mathbb{P}\mathbb{Q}|_{\mathfrak{F}} = \mathbb{F}\mathbb{Q}\mathbb{F}\mathbb{P}|_{\mathfrak{F}}.$$

In addition, two elements X and Y in $\mathscr{E}$ are called $\mathbb{F}$-conditionally independent if and only if the two order-closed Riesz subspaces $\langle X, \mathcal{R}(\mathbb{F})\rangle$ and $\langle Y, \mathcal{R}(\mathbb{F})\rangle$ generated by X and $\mathcal{R}(\mathbb{F})$ and by Y and $\mathcal{R}(\mathbb{F})$ are $\mathbb{F}$-conditionally independent. By the Radon-Nikodým-Douglas-Andô type theorem, established in [24], a subset G of an $\mathbb{F}$-universally Dedekind complete Riesz space $\mathscr{E}$ is a closed Riesz subspace of $\mathscr{E}$ with $\mathcal{R}(\mathbb{F}) \subset G$ if and only if there is a unique conditional expectation $\mathbb{F}_G$ on $\mathscr{E}$ with $\mathcal{R}(\mathbb{F}_G) = G$ and $\mathbb{F}_G\mathbb{F} = \mathbb{F} = \mathbb{F}\mathbb{F}_G$. As a consequence of this theorem, two closed Riesz subspaces $\mathscr{E}_1$ and $\mathscr{E}_2$ with $\mathcal{R}(\mathbb{F}) \subset \mathscr{E}_1 \cap \mathscr{E}_2$

are $\mathbb{F}$-conditionally independent if and only if

$$\mathbb{F}_1\mathbb{F}_2 = \mathbb{F} = \mathbb{F}_2\mathbb{F}_1. \tag{2.2}$$

The equality (2.2) is equivalent to

$$\mathbb{F}_i X = \mathbb{F}X, \qquad \text{for all } X \in \mathscr{E}_{3-i}, \quad i = 1, 2. \tag{2.3}$$

For more information about independence, see [23, 17]. The domain of a conditional expectation $\mathbb{F}$ can be extended to the *natural domain* $\mathcal{L}^1(\mathbb{F})$, which is an $\mathbb{F}$-universally Dedekind complete ideal of the universal completion $\mathscr{E}^u$. This extension, $\mathbb{F}'$, is a conditional expectation on $\mathcal{L}^1(\mathbb{F})$. We shall always identify $\mathbb{F}$ and $\mathbb{F}'$ in this paper without further mention. Note that $\mathcal{L}^1(\mathbb{F})^u = \mathscr{E}^u$. Since $\mathscr{E}^u$ has an f-algebra structure, the multiplication of elements of $\mathscr{E}$ is defined, but it is not necessarily an element of $\mathscr{E}$. This led Labuschagne and Watson to define

$$\mathcal{L}^2(\mathbb{F}) := \{X \in \mathcal{L}^1(\mathbb{F}) \mid X^2 \in \mathcal{L}^1(\mathbb{F})\}.$$

If $X, Y \in \mathcal{L}^2(\mathbb{F})$, then $0 \leq (X \pm Y)^2 = X^2 \pm 2XY + Y^2$. Therefore, $\pm 2XY \leq X^2 + Y^2$ and $2|XY| \leq X^2 + Y^2 \in \mathcal{L}^1(\mathbb{F})$. Hence, $XY \in \mathcal{L}^1(\mathbb{F})$ (see [21]).
Let $\mathbb{F}$ be a conditional expectation on the Riesz space $\mathscr{E}$ with weak order unit E. By Cantelli's inequality in Divandar and Sadeghi (2023), if $X \in \mathscr{E}$, then for every $\lambda > 0$,

$$\mathbb{F}\mathbb{P}_{(|X|-\lambda E)^+}E \leq \frac{\mathbb{F}(|X|^2)}{\lambda^2 + \mathbb{F}(|X|^2)}.$$

This inequality overcomes the rough upper bound which can be obtained by Chebyshev's inequality as follows:

$$\mathbb{F}\mathbb{P}_{(|X|-\lambda E)^+}E \leq \frac{\mathbb{F}(|X|^2)}{\lambda^2},$$

when $\frac{\mathbb{F}(|X|^2)}{\lambda^2} \geq 1$ (see [12]).

Kolmogorov's inequality was generalized from classical probability to Riesz spaces in [6]. Indeed, if $X_1, X_2, \ldots, X_n \in \mathcal{L}^2(\mathbb{F})$ are $\mathbb{F}$-independent and $\mathbb{F}X_k = 0$ for every k, then for $\lambda > 0$, the following inequality holds:

$$\lambda^2\mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n}|S_k|-\lambda E)^+}E \leq \sum_{k=1}^{n}\mathbb{F}{X_k}^2. \tag{2.4}$$

A generalization of Kolmogorov's inequality, called the Hájek-Rényi inequality, was ex-

pressed in [6] as follows.
Let $X_1, X_2, \ldots, X_n \in \mathcal{L}^2(\mathbb{F})$ be $\mathbb{F}$-independent with $\mathbb{F}X_k = 0$ for every k, and let $\{r_k \mid 0 \leq k \leq n\}$ be positive, non-increasing scalars. Then for $\lambda > 0$,

$$\lambda^2 \mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n} |r_k S_k| - \lambda E)^+} E \leq \sum_{k=1}^{n} r_k^2 \mathbb{F}X_k^2. \tag{2.5}$$

If $r_k = 1$ for all k, then the Hájek-Rényi inequality reduces to Kolmogorov's inequality. In this paper, we aim to state some probability inequalities, such as Marshall's inequality, Shao's inequality, etc., in the framework of Riesz spaces.

3 Maximal Inequalities

In this section, some probability inequalities are established in the framework of Riesz spaces. We assume throughout that $\mathbb{F}$ is a conditional expectation on the Dedekind complete Riesz space $\mathscr{E}$ with a weak order unit E, such that $\mathbb{F}(E) = E$. By considering $\mathcal{L}^1(\mathbb{F})$, we will also assume that the space $\mathscr{E}$ is $\mathbb{F}$-universally complete. To simplify, we write $\mathbb{F}$-conditionally independent simply as $\mathbb{F}$-independent. Furthermore, for $1 \leq k \leq n$, we set $S_k := \sum_{i=1}^{k} X_i$. We provide Marshall's inequality, which gives a bound on the probability for the partial sums of a sequence of independent random variables. Note that Marshall's inequality is stronger than Kolmogorov's inequality.

Theorem 3.1. (Marshall's inequality)
Let $X_1, X_2, \ldots, X_n \in \mathcal{L}^2(\mathbb{F})$ be $\mathbb{F}$-independent with $\mathbb{F}X_k = 0$ for every k. Then for $\lambda > 0$,

$$\mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n} |S_k| - \lambda E)^+} E \leq \frac{\sum_{k=1}^{n} \mathbb{F}X_k^2}{\lambda^2 + \sum_{k=1}^{n} \mathbb{F}X_k^2}.$$

Proof . Let $S := \sup_{1\leq k\leq n}|S_k|$. Define

$$B_1 := B(|S_1| \geq \lambda E) = B\left(|S_1| + \frac{\mathbb{F}X_1^2}{\lambda E} \geq \lambda E + \frac{\mathbb{F}X_1^2}{\lambda E}\right),$$

$$B_2 := B(|S_1| < \lambda E, |S_2| \geq \lambda E) = B\left(|S_1| + \frac{\mathbb{F}X_1^2}{\lambda E} < \lambda E + \frac{\mathbb{F}X_1^2}{\lambda E}, |S_2| + \frac{\mathbb{F}X_2^2}{\lambda E} \geq \lambda E + \frac{\mathbb{F}X_2^2}{\lambda E}\right),$$

$$\vdots$$

$$\begin{aligned} B_n &:= B(|S_1| < \lambda E, \ldots, |S_{n-1}| < \lambda E, |S_n| \geq \lambda E) \\ &= B\left(|S_1| + \frac{\mathbb{F}X_1^2}{\lambda E} < \lambda E + \frac{\mathbb{F}X_1^2}{\lambda E}, \ldots, |S_{n-1}| + \frac{\mathbb{F}X_{n-1}^2}{\lambda E} < \lambda E + \frac{\mathbb{F}X_{n-1}^2}{\lambda E},\right. \\ &\qquad \left. |S_n| + \frac{\mathbb{F}X_n^2}{\lambda E} \geq \lambda E + \frac{\mathbb{F}X_n^2}{\lambda E}\right), \end{aligned}$$

with corresponding band projections $\mathbb{P}_{\mathcal{B}_k}$, for $1 \leq k \leq n$. If $\mathbb{P}$ denotes the projection onto the band $\mathcal{B}(S \geq \lambda E) = \mathcal{B}\left(S + \frac{\sum_{k=1}^n \mathbb{F}X_k^2}{\lambda E} \geq \lambda E + \frac{\sum_{k=1}^n \mathbb{F}X_k^2}{\lambda E}\right)$, we have that $\mathbb{P} = \sum_{k=1}^n \mathbb{P}_{\mathcal{B}_k}$. Therefore, we get that

$$\begin{aligned} \mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n}|S_k|-\lambda E)^+}E &= \mathbb{F}\mathbb{P}_{\left(\sup_{1\leq k\leq n}\left(|S_k|+\frac{\sum_{k=1}^n \mathbb{F}X_k^2}{\lambda E}\right)-\left(\lambda E+\frac{\sum_{k=1}^n \mathbb{F}X_k^2}{\lambda E}\right)\right)^+}E \\ &\leq \frac{\sum_{k=1}^n \mathbb{F}\left(X_k + \frac{\sum_{k=1}^n \mathbb{F}X_k{}^2}{\lambda E}\right)^2}{\left(\lambda E + \frac{\sum_{k=1}^n \mathbb{F}X_k^2}{\lambda E}\right)^2} \\ &= \frac{\sum_{k=1}^n \mathbb{F}X_k^2 + \sum_{k=1}^n \frac{\mathbb{F}X_k^4}{\lambda^2} + 2\sum_{k=1}^n \mathbb{F}\left(X_k \frac{\mathbb{F}X_k^2}{\lambda}\right)}{\lambda^{-2}(\lambda^2 + \sum_{k=1}^n \mathbb{F}X_k^2)^2}. \end{aligned}$$

Since $\mathbb{F}X_k = 0$, it implies that

$$\begin{aligned} \mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n}|S_k|-\lambda E)^+}E &\leq \frac{\sum_{k=1}^n \mathbb{F}X_k^2 \lambda^{-2}(\lambda^2 + \sum_{k=1}^n \mathbb{F}X_k^2)}{\lambda^{-2}(\lambda^2 + \sum_{k=1}^n \mathbb{F}X_k^2)^2} \\ &= \frac{\sum_{k=1}^n \mathbb{F}X_k^2}{\lambda^2 + \sum_{k=1}^n \mathbb{F}X_k^2}. \end{aligned}$$

□

In the following theorem, we state an inequality of maximal increment.

Theorem 3.2. (Shao's inequality)
Let $\{X_n\} \in \mathcal{L}^2(\mathbb{F})$ be $\mathbb{F}$-independent, and let there exist $\epsilon > 0$, $0 < \alpha < 1$, and $m \geq 1$ such

that for a certain $\lambda > 0$,

$$\beta = \mathbb{FP}_{(\sup_{1\le k\le m}|S_k|-\epsilon\lambda E)^+}E \le \alpha E.$$

Then

$$\mathbb{F}\sum_{n=0}^{m}\mathbb{P}_{(\sup_{1\le k\le N}|S_{n+k}-S_n|-\lambda E)^-}E \le \frac{E}{E-\alpha}\mathbb{FP}_{(\sup_{1\le k\le N}|S_k|-(1+\epsilon)\lambda E)^-}E.$$

Proof . Let $B_m = B(\lambda E - S_m^*)$, where $S_m^* = \sup_{1\le k\le N}|S_{m+k} - S_m|$. For $i = m-1, m-2, \ldots, 0$, we set

$$B_i = \bigcap_{i<n\le m} B(S^* - \lambda E) \cap B(\lambda E - S_i^*),$$

where $S^* = \sup_{1\le k\le N}|S_{n+k} - S_n|$, and $S_i^* = \sup_{1\le k\le N}|S_{i+k} - S_i|$. Take

$$\begin{aligned} A = B(\sup_{1\le k\le N}|S_k| - (1+\epsilon)\lambda E) &= B(\sup_{1\le k\le n}|S_k| - (1+\epsilon)\lambda E) \\ &\cup B(\sup_{n\le k\le N}|S_k| - (1+\epsilon)\lambda E) \\ &= A_1 \cup A_2. \end{aligned}$$

Then,

$$\begin{aligned} \bigcup_{n=0}^{m} B_n &= (\bigcup_{n=0}^{m} B_n \cap A^d) \bigcup (\bigcup_{n=0}^{m} B_n \cap A) \\ &\subset A^d \cup (\bigcup_{n=0}^{m} B_n \cap A_1) \cup (\bigcup_{n=0}^{m} B_n \cap A_2). \end{aligned} \tag{3.1}$$

Now let $C = B(|S_n| - \epsilon\lambda E)$; then $A_2 \subset C$. If $D = B(\sup_{1\le k\le n}|S_k| - \epsilon\lambda E)$, then $A_1 \subseteq D$. Therefore, by (3.1), we obtain

$$\begin{aligned} \bigcup_{n=0}^{m} B_n &\subset A^d \cup (\bigcup_{n=0}^{m} B_n \cap A_1) \cup (\bigcup_{n=0}^{m} B_n \cap C) \\ &\subset A^d \cup (\bigcup_{n=0}^{m} B_n \cap D). \end{aligned}$$

Now, by $\mathbb{F}$-independence, we get that

$$\begin{aligned}
\mathbb{F}\sum_{n=0}^{m}\mathbb{P}_{(\sup_{1\le k\le N}|S_{n+k}-S_n|-\lambda E)^+}E &= \mathbb{F}\mathbb{P}_{\bigcup_{n=0}^{m}B_n}E\\
&\le \mathbb{F}\mathbb{P}_{A^d\cup(\bigcup_{n=0}^{m}B_n\cap D)}E\\
&= \mathbb{F}\mathbb{P}_{(\sup_{1\le k\le N}|S_k|-(1+\epsilon)\lambda E)^+}E\\
&+ \mathbb{F}\sum_{n=0}^{m}\mathbb{P}_{(\sup_{1\le k\le N}|S_{n+k}-S_n|-\lambda E)^+}E\cdot\mathbb{F}\mathbb{P}_{(\sup_{1\le k\le n}|S_k|-\epsilon\lambda E)^+}E\\
&\le \mathbb{F}\mathbb{P}_{(\sup_{1\le k\le N}|S_k|-(1+\epsilon)\lambda E)^+}E+\alpha\mathbb{F}\sum_{n=0}^{m}\mathbb{P}_{(\sup_{1\le k\le N}|S_{n+k}-S_n|-\lambda E)^+}E.
\end{aligned}$$

Therefore,

$$(E-\alpha)\mathbb{F}\sum_{n=0}^{m}\mathbb{P}_{(\sup_{1\le k\le N}|S_{n+k}-S_n|-\lambda E)^+}E\le\mathbb{F}\mathbb{P}_{(\sup_{1\le k\le N}|S_k|-(1+\epsilon)\lambda E)^+}E,$$

and

$$\mathbb{F}\sum_{n=0}^{m}\mathbb{P}_{(\sup_{1\le k\le N}|S_{n+k}-S_n|-\lambda E)^+}E\le\frac{E}{E-\alpha}\mathbb{F}\mathbb{P}_{(\sup_{1\le k\le N}|S_k|-(1+\epsilon)\lambda E)^+}E.$$

□

In Etemadi's inequality, which is stated in [6], it is proven that if $X_1, X_2, \ldots, X_n$ are $\mathbb{F}$-independent elements of an $\mathbb{F}$-universally Dedekind complete Riesz space $\mathscr{E}$ with a weak order unit E, then for $\lambda > 0$ and $1 \le k \le n$,

$$\mathbb{F}\mathbb{P}_{(\sup_{1\le k\le n}|S_k|-3\lambda E)^+}E\le 3\sup_{1\le k\le n}\mathbb{F}\mathbb{P}_{(|S_k|-\lambda E)^+}E. \tag{3.2}$$

In the following theorem, we find a better upper bound for Etemadi's inequality.

Theorem 3.3. Let $X_1, X_2, \ldots, X_n \in \mathcal{L}^2(\mathbb{F})$ be $\mathbb{F}$-independent with $\mathbb{F}X_k = 0$ for every k. Then for $\lambda > 0$,

$$\mathbb{F}\mathbb{P}_{(\sup_{1\le k\le n}|S_k|-3\lambda E)^+}E\le 2\sup_{1\le k\le n}\mathbb{F}\mathbb{P}_{(|S_k|-\lambda E)^+}E.$$

Proof . Let $\nu, \lambda \geq 0$ be arbitrary positive vectors and define

$$\begin{aligned}
&B_1 := \mathcal{B}(|S_n - S_{n-1}| > (\nu + \lambda)E),\\
&B_2 := \mathcal{B}(|S_n - S_{n-1}| \leq (\nu + \lambda)E, |S_n - S_{n-2}| > (\nu + \lambda)E),\\
&\vdots\\
&B_{n-1} := \mathcal{B}(|S_n - S_{n-1}| \leq (\nu + \lambda)E, \ldots, |S_n - S_2| \leq (\nu + \lambda)E, |S_n - S_1| > (\nu + \lambda)E),\\
&B_n := \mathcal{B}(|S_n - S_{n-1}| \leq (\nu + \lambda)E, \ldots, |S_n - S_1| \leq (\nu + \lambda)E, |S_n - S_0| > (\nu + \lambda)E),
\end{aligned}$$

with corresponding band projections $\mathbb{P}_{B_k}$, for $1 \leq k \leq n$. Note that

$$\sup_{1\leq k\leq n} |S_n - S_{n-k}| = \sup_{0\leq k<n} |S_n - S_k|,$$

thus

$$B(\sup_{1\leq k\leq n} |S_n - S_{n-k}| > (\nu + \lambda)E) = B(\sup_{0\leq k<n} |S_n - S_k| > (\nu + \lambda)E).$$

If $\mathbb{P}$ denotes the projection onto the band $\mathcal{B}(\sup_{1\leq k\leq n} |S_n - S_{n-k}| > (\nu + \lambda)E)$, then $\mathbb{P} = \sum_{k=1}^{n} \mathbb{P}_{B_k}$, and we obtain

$$\begin{aligned}
\mathbb{F}\mathbb{P}_{(\sup_{0\leq k<n} |S_n - S_k| - (\nu+\lambda)E)^+ \cap \mathcal{B}(|S_n|-\lambda E)^+} E &= \mathbb{F} \sum_{k=1}^{n} \mathbb{P}_{B_k} \mathbb{P}_{\mathcal{B}(|S_n|-\lambda E)^+} E\\
&= \sum_{k=1}^{n} \mathbb{F}\mathbb{P}_{B_k} \mathbb{P}_{\mathcal{B}(|S_n|-\lambda E)^+} E.
\end{aligned}$$

Now suppose that $|S_n - S_{n-k}| > (\nu + \lambda)E$ and $|S_n| \leq \lambda E$; then we get that $|S_{n-k}| > \nu E$, hence

$$B(|S_n - S_{n-k}| > (\nu + \lambda)E) \cap B(|S_n| \leq \lambda E) \subseteq B(|S_{n-k}| > \nu E). \tag{3.3}$$

Therefore, we obtain

$$\begin{aligned}
\mathbb{F}\mathbb{P}_{(\sup_{0\leq k<n} |S_n - S_k| - (\nu+\lambda)E)^+ \cap \mathcal{B}(|S_n|-\lambda E)^+} E &\leq \sum_{k=1}^{n} \mathbb{F}\mathbb{P}_{B_k} \mathbb{F}\mathbb{P}_{\mathcal{B}(|S_{n-k}|-\nu E)^+} E\\
&\leq \mathbb{F}\mathbb{P}_{(\sup_{0\leq k<n} |S_n - S_k| - (\nu+\lambda)E)^+} E\\
&\quad \cdot \sup_{1\leq k\leq n} \mathbb{F}\mathbb{P}_{\mathcal{B}(|S_{n-k}|-\nu E)^+} E.
\end{aligned}$$

Thus, we get that

$$\mathbb{FP}_{(\sup_{0\leq k<n}|S_n-S_k|-(\nu+\lambda)E)^+\cap\mathcal{B}(|S_n|-\lambda E)^+}E \leq \sup_{0\leq k<n}\mathbb{FP}_{\mathcal{B}(|S_k|-\nu E)^+}E \cdot \mathbb{FP}_{(\sup_{0\leq k<n}|S_n-S_k|-(\nu+\lambda)E)^+}E. \tag{3.4}$$

Note that

$$\mathcal{B}(\sup_{1\leq k\leq n}|S_k| > (\nu+2\lambda)E) \subseteq \mathcal{B}(|S_n| > \lambda E) \cup (\mathcal{B}(\sup_{0\leq k<n}|S_n - S_k| > (\nu+\lambda)E) \cap \mathcal{B}(|S_n| \leq \lambda E)). \tag{3.5}$$

Thus, by (3.4) and (3.5), we obtain

$$\begin{aligned}\mathbb{FP}_{(\sup_{1\leq k\leq n}|S_k|-(\nu+2\lambda)E)^+}E &\leq \mathbb{FP}_{\mathcal{B}(|S_n|-\lambda E)^+}E \\ &\quad + \mathbb{FP}_{(\sup_{0\leq k<n}|S_n-S_k|-(\nu+\lambda)E)^+}E \cdot \mathbb{FP}_{\mathcal{B}(|S_n|-\lambda E)^-}E \\ &\leq \mathbb{FP}_{\mathcal{B}(|S_n|-\lambda E)^+}E + \sup_{0\leq k<n}\mathbb{FP}_{\mathcal{B}(|S_k|-\nu E)^+}E \\ &\quad \cdot \mathbb{FP}_{(\sup_{0\leq k<n}|S_n-S_k|-(\nu+\lambda)E)^+}E.\end{aligned}$$

Since $\mathbb{FP}_{(\sup_{0\leq k<n}|S_n-S_k|-(\nu+\lambda)E)^+}E \leq \mathbb{F}E = E$, it implies that

$$\mathbb{FP}_{(\sup_{1\leq k\leq n}|S_k|-(\nu+2\lambda)E)^+}E \leq \mathbb{FP}_{\mathcal{B}(|S_n|-\lambda E)^+}E + \sup_{0\leq k<n}\mathbb{FP}_{\mathcal{B}(|S_k|-\nu E)^+}E.$$

Putting $\nu = \lambda$, we get

$$\mathbb{FP}_{(\sup_{1\leq k\leq n}|S_k|-3\lambda E)^+}E \leq 2\sup_{1\leq k\leq n}\mathbb{FP}_{\mathcal{B}(|S_k|-\lambda E)^+}E.$$

□

By [6], we recall that two random variables X and Y are called $\mathbb{F}$-identically distributed whenever

$$f_X(t) = f_Y(t) \quad (t \in \mathbb{R}),$$

in which the $\mathbb{F}$-distribution function $f_X : \mathbb{R} \to [0, E]$ is defined by [3] as follows:

$$f_X(t) = E - \mathbb{FP}_{(X-tE)^+}E.$$

A random variable $X \in \mathscr{E}$ is called $\mathbb{F}$-symmetric if X and $-X$ are $\mathbb{F}$-identically distributed. The median, $\mathrm{med}(X)$, of a random variable X is a real number satisfying

$$\mathbb{FP}_{(X-\mathrm{med}(X)E)^+}E \geq \frac{1}{2}E, \tag{3.6}$$

and

$$\mathbb{FP}_{(X-\mathrm{med}(X)E)^-}E \geq \frac{1}{2}E. \tag{3.7}$$

In fact, a median is a type of center of the distribution. It means that (at least) half of the probability mass lies to the left of it and (at least) half of it to the right. Sometimes, to prove certain theorems, we need the condition of symmetrization for random variables even when they are not symmetric. In these cases, we use the concept of a symmetrized random variable.

Definition 3.4. Let X and Y be $\mathbb{F}$-independent and $\mathbb{F}$-identically distributed elements of $\mathscr{E}$. We call $Z = X - Y$ the symmetrized random variable.

In the following results, we state the relation between the tail probabilities of random variables and the tail probabilities of their symmetrizations.

Theorem 3.5 (Weak symmetrization inequalities). Let X and Y be $\mathbb{F}$-independent and $\mathbb{F}$-identically distributed elements of $\mathscr{E}$. For every positive λ and μ,

$$\mathbb{FP}_{(X-\mathrm{med}(X)E-\lambda E)^+}E \leq 2\mathbb{FP}_{(Z-\lambda E)^+}E, \tag{3.8}$$

and

$$\mathbb{FP}_{(|X-\mathrm{med}(X)E|-\lambda E)^+}E \leq \mathbb{FP}_{(|Z|-\lambda E)^+}E \leq 2\mathbb{FP}_{\left(|X-\mu E|-\frac{\lambda}{2}E\right)^+}E. \tag{3.9}$$

Proof . Since X and Y are $\mathbb{F}$-identically distributed, we have

$$f_X(t) = f_Y(t) \quad (t \in \mathbb{R}).$$

Hence, by (3.6) and (3.7), we get that $\mathrm{med}(X) = \mathrm{med}(Y)$. For the symmetrized random variable Z, we have

$$Z = X - Y = (X - \mathrm{med}(X)E) - (Y - \mathrm{med}(Y)E).$$

We claim that

$$\mathcal{B}(X - \mathrm{med}(X)E - \lambda E)^+ \cap \mathcal{B}(\mathrm{med}(Y)E - Y)^+ \subseteq \mathcal{B}(Z - \lambda E)^+. \tag{3.10}$$

To prove the claim, suppose that

$$W \in \mathcal{B}(X - \mathrm{med}(X)E - \lambda E)^+ \cap \mathcal{B}(\mathrm{med}(Y)E - Y)^+.$$

Hence,

$$|W| = \sup_n |W| \wedge n|X - \mathrm{med}(X)E - \lambda E| \quad \text{and} \quad |W| = \sup_n |W| \wedge n|\mathrm{med}(Y)E - Y|.$$

Therefore,

$$\begin{aligned} |W| &= \sup_n |W| \wedge n|X - \mathrm{med}(X)E - \lambda E| \wedge n|\mathrm{med}(Y)E - Y| \\ &\leq \sup_n |W| \wedge n|X - \mathrm{med}(X)E - \lambda E + \mathrm{med}(Y)E - Y| \\ &= \sup_n |W| \wedge n|Z - \lambda E|. \end{aligned}$$

Thus, $W \in \mathcal{B}(Z - \lambda E)^+$. By (3.10), we get that

$$\mathbb{FP}_{(Z-\lambda E)^+}E \geq \mathbb{FP}_{(X-\mathrm{med}(X)E-\lambda E)^+}E \cdot \mathbb{FP}_{(\mathrm{med}(Y)E-Y)^+}E.$$

By the definition of $\mathrm{med}(Y)$, we obtain that

$$\mathbb{FP}_{(Z-\lambda E)^+}E \geq \frac{1}{2}\mathbb{FP}_{(X-\mathrm{med}(X)E-\lambda E)^+}E,$$

which proves the inequality (3.8). The first inequality in (3.9) is proved in a similar way by replacing X with $-X$. Now, we prove the second inequality in (3.9) as follows. Suppose that λ and μ are positive; then we get that

$$|Z| - \lambda E = \left(|X - \mu E| - \frac{\lambda}{2}E\right) + \left(|Y - \mu E| - \frac{\lambda}{2}E\right).$$

It means that

$$\mathbb{FP}_{(|Z|-\lambda E)^+}E \leq \mathbb{FP}_{\left(|X-\mu E|-\frac{\lambda}{2}E\right)^+}E + \mathbb{FP}_{\left(|Y-\mu E|-\frac{\lambda}{2}E\right)^+}E.$$

Since X and Y are $\mathbb{F}$-identically distributed, we have

$$\mathbb{FP}_{\left(|X-\mu E|-\frac{\lambda}{2}E\right)^+}E = \mathbb{FP}_{\left(|Y-\mu E|-\frac{\lambda}{2}E\right)^+}E.$$

Therefore,

$$\mathbb{FP}_{(|Z|-\lambda E)^+}E \leq 2\mathbb{FP}_{\left(|X-\mu E|-\frac{\lambda}{2}E\right)^+}E.$$

□

Remark 3.6. In Theorem 3.5, if X and Y are $\mathbb{F}$-independent and $\mathbb{F}$-symmetric, then

$$\mathbb{FP}_{(|X-\mathrm{med}(X)E|-\lambda E)^+}E \leq \mathbb{FP}_{(|Z|-\lambda E)^+}E \leq 2\mathbb{FP}_{\left(|X-\mathrm{med}(X)E|-\frac{\lambda}{2}E\right)^+}E. \tag{3.11}$$

To prove it, it is sufficient to replace μ with $\mathrm{med}(X)$ in inequality (3.9).

In the following theorem, we extend the weak symmetrization inequalities to maximal inequalities.

Theorem 3.7. (Strong symmetrization inequalities)
Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be $\mathbb{F}$-independent elements of an $\mathbb{F}$-universally Dedekind complete Riesz space $\mathscr{E}$ in which X_k and Y_k are $\mathbb{F}$-identically distributed for $1 \leq k \leq n$. Take $Z_k = X_k - Y_k$; then for $\lambda > 0$ and positive sequences $\{\mu_k \mid 1 \leq k \leq n\}$,

$$\mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n}(X_k - \mathrm{med}(X_k)E) - \lambda E)^+} E \leq 2\mathbb{F}\mathbb{P}_{(\sup_{1\leq k\leq n} Z_k - \lambda E)^+} E. \tag{3.12}$$

Proof . Suppose that

$$\begin{aligned}
\mathcal{B}_1 &:= \mathcal{B}(X_1 - \mathrm{med}(X_1)E \geq \lambda E),\\
\mathcal{B}_2 &:= \mathcal{B}(X_1 - \mathrm{med}(X_1)E < \lambda E, X_2 - \mathrm{med}(X_2)E \geq \lambda E),\\
&\vdots\\
\mathcal{B}_n &:= \mathcal{B}(X_1 - \mathrm{med}(X_1)E < \lambda E, \ldots, X_{n-1} - \mathrm{med}(X_{n-1})E < \lambda E,\\
&\qquad X_n - \mathrm{med}(X_n)E \geq \lambda E),
\end{aligned}$$

with corresponding band projections $\mathbb{P}_{\mathcal{B}_k}$, for $1 \leq k \leq n$. If $\mathbb{P}$ denotes the projection onto the band $\mathcal{B}(\sup_{1\leq k\leq n}(X_k - \mathrm{med}(X_k)E) - \lambda E)^+$, we have that $\mathbb{P} = \sum_{k=1}^n \mathbb{P}_{\mathcal{B}_k}$. For $1 \leq k \leq n$, put $\mathcal{A}_k := \mathcal{B}(Y_k - \mathrm{med}(Y_k)E)^-$. Furthermore, define

$$\begin{aligned}
C_1 &:= \mathcal{B}(Z_1 \geq \lambda E),\\
C_2 &:= \mathcal{B}(Z_1 < \lambda E, Z_2 \geq \lambda E),\\
&\vdots\\
C_n &:= \mathcal{B}(Z_1 < \lambda E, \ldots, Z_{n-1} < \lambda E, Z_n \geq \lambda E),
\end{aligned}$$

with corresponding band projections $\mathbb{P}_{C_k}$, for $1 \leq k \leq n$. If $\mathbb{P}'$ denotes the projection onto the band $\mathcal{B}(\sup_{1\leq k\leq n} Z_k - \lambda E)^+$, we have that $\mathbb{P}' = \sum_{k=1}^n \mathbb{P}_{C_k}$.

Now we get that $\mathcal{A}_k \cap \mathcal{B}_k \subseteq C_k$ for $1 \leq k \leq n$ since

$$\begin{aligned}
Z_k &= X_k - Y_k - \mathrm{med}(X_k)E + \mathrm{med}(X_k)E\\
&= X_k - \mathrm{med}(X_k)E + \mathrm{med}(Y_k)E - Y_k\\
&\geq \lambda E.
\end{aligned}$$

Therefore, we obtain $\mathbb{P}_{\mathcal{A}_k\cap\mathcal{B}_k}E \le \mathbb{P}_{C_k}E$, hence

$$\begin{aligned}\mathbb{FP}_{\left(\sup_{1\le k\le n} Z_k-\lambda E\right)^+}E &= \sum_{k=1}^{n}\mathbb{FP}_{C_k}E\\ &\ge \sum_{k=1}^{n}\mathbb{F}\left(\mathbb{P}_{\mathcal{A}_k}\mathbb{P}_{\mathcal{B}_k}\right)E\\ &= \sum_{k=1}^{n}\left(\mathbb{FP}_{\mathcal{A}_k}E\right)\cdot\left(\mathbb{FP}_{\mathcal{B}_k}E\right)\\ &\ge \frac{1}{2}\left(\sum_{k=1}^{n}\mathbb{FP}_{\mathcal{B}_k}E\right)\\ &= \frac{1}{2}\mathbb{FP}_{\left(\sup_{1\le k\le n}(X_k-\mathrm{med}(X_k)E)-\lambda E\right)^+}E. \end{aligned} \tag{3.13}$$

□

Corollary 3.8. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be $\mathbb{F}$-independent elements of an $\mathbb{F}$-universally Dedekind complete Riesz space $\mathscr{E}$ in which X_k and Y_k are $\mathbb{F}$-identically distributed for $1 \le k \le n$. Take $Z_k = X_k - Y_k$; then for $\lambda > 0$ and positive sequences $\{\mu_k \mid 1 \le k \le n\}$,

$$\begin{aligned}\mathbb{FP}_{(\sup_{1\le k\le n}|X_k-\mathrm{med}(X_k)E|-\lambda E)^+}E &\le \mathbb{FP}_{(\sup_{1\le k\le n}|Z_k|-\lambda E)^+}E\\ &\le 2\mathbb{FP}_{\left(\sup_{1\le k\le n}|X_k-\mu_k E|-\frac{\lambda}{2}E\right)^+}E. \end{aligned} \tag{3.14}$$

In particular, for symmetric cases, we get that

$$\begin{aligned}\mathbb{FP}_{(\sup_{1\le k\le n}|X_k-\mathrm{med}(X_k)E|-\lambda E)^+}E &\le \mathbb{FP}_{(\sup_{1\le k\le n}|Z_k|-\lambda E)^+}E\\ &\le 2\mathbb{FP}_{\left(\sup_{1\le k\le n}|X_k-\mathrm{med}(X_k)E|-\frac{\lambda}{2}E\right)^+}E. \end{aligned} \tag{3.15}$$

Acknowledgement
The authors would like to express their deepest gratitude to the anonymous referees for their useful comments.

Conflict of interest
The authors declare that they have no conflict of interest.

References

[1] C. D. Aliprantis, and O. Burkinshaw, *Positive Operators*, Academic Press, Orlando, 1985.
zbMATH: 0608.47039 **MR:** 0809372 **DOI:** 10.1007/978-1-4020-5008-4

[2] Y. Azouzi, and K. Ramdane, *Burkholder inequalities in Riesz spaces*, Indag. Math., **28** (2017) 1076–1094.
zbMATH: 1386.60156 **MR:** 3697048 **DOI:** 10.1016/j.indag.2017.07.007

[3] Y. Azouzi, and K. Ramdane, *On the distribution function with respect to conditional expectation on Riesz spaces*, Quaest. Math., **41** (2017) 257–264.
zbMATH: 1396.60005 **MR:** 3777886 **DOI:** 10.2989/16073606.2017.1377310

[4] K. Boulabiar, A. Buskes, and A. Triki, *Results in f-algebras*, Positivity, Trends in Mathematics, Birkhäuser, Basel, 2007, pp. 73–96.
zbMATH: 1149.06009 **MR:** 2382215 **DOI:** 10.1007/978-3-7643-8478-4_3

[5] G. Buskes, and A. van Rooij, *Almost f-algebras: Commutativity and the Cauchy-Schwarz inequality*, Positivity, **4** (2000) 227–231.
zbMATH: 0987.46002 **MR:** 1797125 **DOI:** 10.1023/A:1009826510957

[6] M. S. Divandar, and Gh. Sadeghi, *Maximal probability inequalities in vector lattices*, Results Math., **77** (2022) 1–18.
zbMATH: 1496.46002 **MR:** 4404440 **DOI:** 10.1007/s00025-022-01637-0

[7] M. S. Divandar, and Gh. Sadeghi, *The Itô integral and near-martingales in Riesz spaces*, Comm. Statist. Theory Methods, **52** (2023) 5068–5081.
zbMATH: 07710583 **MR:** 4593857 **DOI:** 10.1080/03610926.2021.2003401

[8] H. E. Gessesse, and V. G. Troitsky, *Martingales in Banach lattices II*, Positivity, **15** (2011) 49–55.
zbMATH: 1218.60035 **MR:** 2782746 **DOI:** 10.1007/s11117-009-0040-5

[9] J. J. Grobler, *Continuous stochastic processes in Riesz spaces: the Doob-Meyer decomposition*, Positivity, **14** (2010) 731–751.
zbMATH: 1216.46005 **MR:** 2741330 **DOI:** 10.1007/s11117-010-0088-2

[10] J. J. Grobler, *Doob's optional sampling theorem in Riesz spaces*, Positivity, **15** (2011) 617–637.
zbMATH: 1242.60041 **MR:** 2861602 **DOI:** 10.1007/s11117-011-0114-z

[11] J. J. Grobler, *Jensen's and martingale inequalities in Riesz spaces*, Indag. Math., **24** (2014) 275–295.
zbMATH: 1307.47095 **MR:** 3151817 **DOI:** 10.1016/j.indag.2013.02.003

[12] J. J. Grobler, *The Kolmogorov-Centsov theorem and Brownian motion in vector lattices*, J. Math. Anal. Appl., **410** (2014) 891–901.
zbMATH: 1335.60155 **MR:** 3111875 **DOI:** 10.1016/j.jmaa.2013.08.056

[13] J. J. Grobler, *Stopped processes and Doob's optional sampling theorem*, J. Math. Anal. Appl., **497** (2021) 124875.
zbMATH: 1464.60037 **MR:** 4192217 **DOI:** 10.1016/j.jmaa.2020.124875

[14] E. Krajewska, *On (in)dependence measures in Riesz spaces*, J. Math. Anal. Appl., **491** (2020) 124266.
zbMATH: 1519.47039 **MR:** 4106725 **DOI:** 10.1016/j.jmaa.2020.124266

[15] W.-Ch. Kuo, C. C. A. Labuschagne, and B. A. Watson, *Discrete time stochastic processes on Riesz spaces*, Indag. Math., **15** (2004) 435–451.
zbMATH: 1057.60041 **MR:** 2093170 **DOI:** 10.1016/S0019-3577(04)80010-7

[16] W.-Ch. Kuo, C. C. A. Labuschagne, and B. A. Watson, *Conditional expectation on Riesz spaces*, J. Math. Anal. Appl., **303** (2005) 509–521.
zbMATH: 1075.46002 **MR:** 2122234 **DOI:** 10.1016/j.jmaa.2004.08.050

[17] W.-Ch. Kuo, C. C. A. Labuschagne, and B. A. Watson, *Ergodic theory and the strong law of large numbers on Riesz spaces*, J. Math. Anal. Appl., **325** (2007) 422–437.
zbMATH: 1114.46015 **MR:** 2273535 **DOI:** 10.1016/j.jmaa.2006.01.056

[18] W.-Ch. Kuo, M. J. Rogans, and B. A. Watson, *Mixing inequalities in Riesz spaces*, J. Math. Anal. Appl., **456** (2017) 992–1004.
zbMATH: 1386.60128 **MR:** 3688462 **DOI:** 10.1016/j.jmaa.2017.07.035

[19] W.-Ch. Kuo, M. J. Rogans, and B. A. Watson, *Near-epoch dependence in Riesz spaces*, J. Math. Anal. Appl., **467** (2017) 462–479.
zbMATH: 1405.60057 **MR:** 3834816 **DOI:** 10.1016/j.jmaa.2018.07.019

[20] W.-Ch. Kuo, D. F. Rodda, and B. A. Watson, *The Hájek-Rényi-Chow maximal inequality and a strong law of large numbers in Riesz spaces*, J. Math. Anal. Appl., **481** (2020) 123462.
zbMATH: 1436.60045 **MR:** 4008549 **DOI:** 10.1016/j.jmaa.2019.123462

[21] C. C. A. Labuschagne, and B. A. Watson, *Discrete stochastic integration in Riesz spaces*, Positivity, **14** (2010) 859–875.
zbMATH: 1220.47054 **MR:** 2741339 **DOI:** 10.1007/s11117-010-0089-1

[22] V. G. Troitsky, *Martingales in Banach lattices*, Positivity, **9** (2005) 437–456.
zbMATH: 1101.60028 **MR:** 2188530 **DOI:** 10.1007/s11117-004-2769-1

[23] J. J. Vardy, and B. A. Watson, *Markov processes on Riesz spaces*, Positivity, **16**

(2012) 373–391.
zbMATH: 1270.47035 **MR:** 2929096 **DOI:** 10.1007/s11117-011-0121-0

[24] B. A. Watson, *An Andô-Douglas type theorem in Riesz spaces with a conditional expectation*, Positivity, **13** (2009) 543–558.
zbMATH: 1186.47038 **MR:** 2511240 **DOI:** 10.1007/s11117-008-2239-2

[25] A. C. Zaanen, *Introduction to Operator Theory in Riesz Spaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1997.
zbMATH: 0878.47022 **MR:** 1631533 **DOI:** 10.1007/978-3-642-60637-3

How to cite this article
M.S. Divandar, Gh. Sadeghi, Some probability inequalities in Riesz spaces, J. Frame Matrix Theory, Vol. 3 (2025) No. 1, 20-36. http://dx.doi.org/10.22034/jfmt.2026.558228.1031.