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Localization of product rings and modules: extension, contraction, and ϕ-primality

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Abstract

Let R_i be commutative rings, S_i multiplicatively closed subsets of R_i, and M_i unitary R_i-modules ($i = 1, 2$). We study the localization of the product ring $R_1 \times R_2$ and the product module $M_1 \times M_2$ at $S_1 \times S_2$. We describe the localization map, its kernel and saturation, and establish the natural isomorphisms with the products of the factor localizations. We then examine extension and contraction of ideals and submodules, exactness, quotient and colon formulas, and preservation of multiplication modules. Finally, using contraction and saturation, we define the induced notion of ϕ-primality and prove transfer and converse results for ideals and submodules. Examples clarify the role of the hypotheses, including finite generation in the colon formula and the properness condition in the transfer theorem.

Introduction

Localization is a basic tool in commutative algebra; it records the part of a ring or module that remains after a multiplicatively closed set is inverted. We consider this

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construction for a product ring $R_1 \times R_2$ and its product module $M_1 \times M_2$, localized at $S_1 \times S_2$, where each S_i is a multiplicatively closed subset of R_i.

The general theory of localization applies to this setting (see, in particular, [2, 5, 9, 4, 10]). The product structure nevertheless permits explicit descriptions of the localization map, its kernel and saturation, and the interaction of extension and contraction with the two factors. We also investigate how localization affects module-theoretic properties and generalized primality.

The paper has two main themes. First, we establish structural results for the localized product ring and module, including exactness, quotient and colon formulas, and preservation of multiplication modules. Second, we define the induced ϕ-operation through contraction and saturation and prove transfer results for ϕ-prime ideals and submodules, together with a converse under suitable hypotheses. Examples indicate where the assumptions are needed.

The organization is as follows. Section 1 fixes notation. Section 2 studies the localized product ring, its saturation, and extension ideals. Section 3 treats the localized product module and its interaction with submodules. Section 4 establishes the results on ϕ-primality. The final section records conclusions and questions.

1 Preliminaries

Throughout, rings are commutative with non-zero identity and modules are unitary. Let R_i be rings, let M_i be R_i-modules, and let $S_i \subseteq R_i$ be multiplicatively closed subsets with $1 \in S_i$ and $0 \notin S_i$ $(i = 1, 2)$. We use the standard construction of ring and module localization; see [10, Section 10.9], [2, Chapter 3], [5, Chapter 2], or [4].

For a multiplicatively closed subset S of a ring R and an R-module M, two pairs $(r, s), (r', s') \in R \times S$ (respectively, $(m, s), (m', s') \in M \times S$) define the same fraction if and only if there is $u \in S$ such that

$$u(rs' - r's) = 0 \quad \text{(respectively, } u(s'm - sm') = 0\text{)}.$$

The usual operations make $S^{-1}R$ a ring and $S^{-1}M$ an $S^{-1}R$-module. In particular,

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'}, \qquad \frac{r}{s}\frac{r'}{s'} = \frac{rr'}{ss'}, \qquad \frac{m}{s} + \frac{m'}{s'} = \frac{ms' + m's}{ss'}.$$

We apply this construction to $R_1 \times R_2$, $M_1 \times M_2$, and $S_1 \times S_2$, and use the notation

$$(S_1 \times S_2)^{-1}(R_1 \times R_2), \qquad (S_1 \times S_2)^{-1}(M_1 \times M_2)$$

for the resulting localization ring and module. Fractions are written coordinatewise, for example

$$\frac{(r_1, r_2)}{(s_1, s_2)}, \qquad \frac{(m_1, m_2)}{(s_1, s_2)}.$$

The natural maps send (r_1, r_2) and (m_1, m_2) to the corresponding fractions with denominator $(1, 1)$. All further notation concerning extension, contraction, saturation, and colon submodules will be specified when it is used.

2 Properties of the ring $(S_1 \times S_2)^{-1}(R_1 \times R_2)$

In this section, $S_1 \subseteq R_1$ and $S_2 \subseteq R_2$ are multiplicatively closed subsets with $0 \notin S_i$ for $i = 1, 2$. We keep the notation of Section 1. In particular,

$$\iota : R_1 \times R_2 \longrightarrow (S_1 \times S_2)^{-1}(R_1 \times R_2), \qquad \iota(r_1, r_2) = \frac{(r_1, r_2)}{(1, 1)},$$

denotes the natural ring homomorphism, and

$$\iota_i : R_i \longrightarrow S_i^{-1} R_i, \qquad \iota_i(r_i) = \frac{r_i}{1},$$

for $i = 1, 2$.

Relation to the general theory. The universal property and the characterization of localization maps used below are standard; see [10, Proposition 10.9.3] and [4, Theorems 3.1 and 4.1]. We state them in product notation but omit routine proofs. The product-specific content of this section is the coordinatewise isomorphism in Theorem 2.3, the saturation formula in Proposition 2.6, and the explicit extension descriptions in Proposition 2.7 and Corollary 2.8.

Proposition 2.1 (Universal property). Let R' be a commutative ring and let

$$g : R_1 \times R_2 \longrightarrow R'$$

be a ring homomorphism such that $g(s_1, s_2)$ is a unit in R' for every $(s_1, s_2) \in S_1 \times S_2$. Then there exists a unique ring homomorphism

$$h : (S_1 \times S_2)^{-1}(R_1 \times R_2) \longrightarrow R'$$

such that $h \circ \iota = g$.

Proof . This is the universal property of localization; see [10, Proposition 10.9.3]. □

For later use, we record the kernel of ι. We have

$$\begin{aligned}\ker \iota &= \left\{(r_1, r_2) \in R_1 \times R_2 : \frac{(r_1, r_2)}{(1,1)} = \frac{(0,0)}{(1,1)}\right\} \\ &= \{(r_1, r_2) \in R_1 \times R_2 : \exists u_1 \in S_1,\ \exists u_2 \in S_2 \text{ such that } u_1 r_1 = 0,\ u_2 r_2 = 0\} \\ &= \{(r_1, r_2) \in R_1 \times R_2 : r_1 \in \ker \iota_1,\ r_2 \in \ker \iota_2\} \\ &= \ker \iota_1 \times \ker \iota_2.\end{aligned}$$

Proposition 2.2. Let R' be a commutative ring and let $g : R_1 \times R_2 \longrightarrow R'$ be a ring homomorphism such that:

(1) $g(s_1, s_2)$ is a unit in R' for every $(s_1, s_2) \in S_1 \times S_2$;
(2) $\ker g = \ker \iota$;
(3) every element of R' is of the form $g(a_1, a_2)g(s_1, s_2)^{-1}$ for some $(a_1, a_2) \in R_1 \times R_2$ and $(s_1, s_2) \in S_1 \times S_2$.

Then the unique ring homomorphism $h : (S_1 \times S_2)^{-1}(R_1 \times R_2) \longrightarrow R'$ with $h \circ \iota = g$, whose existence is guaranteed by Proposition 2.1, is an isomorphism.

Proof . This is the standard characterization of a full localization homomorphism; see [4, Theorem 4.1]. The hypotheses above are precisely its specialization to $S_1 \times S_2$. □

Remark. In the terminology of [4], conditions (1)–(3) of Proposition 2.2 state precisely that g is a *full localization homomorphism* at the localizable set $S_1 \times S_2$. Proposition 2.2 is therefore the specialization, to the product setting, of the general characterization of localization maps in [4].

Theorem 2.3 (Localization of a product). Let R_i be commutative rings and let $S_i \subseteq R_i$ be multiplicatively closed subsets with $0 \notin S_i$, for $i = 1, 2$. Then the map

$$\Phi : (S_1 \times S_2)^{-1}(R_1 \times R_2) \longrightarrow S_1^{-1}R_1 \times S_2^{-1}R_2, \qquad \frac{(r_1, r_2)}{(s_1, s_2)} \longmapsto \left(\frac{r_1}{s_1}, \frac{r_2}{s_2}\right),$$

is a ring isomorphism.

Proof . The coordinatewise equivalence relation defining the localization shows that Φ is well-defined. The formulas for addition and multiplication show that it is a ring homomorphism. It is surjective because every pair $(r_1/s_1, r_2/s_2)$ is the image of $(r_1, r_2)/(s_1, s_2)$. If $\Phi((r_1, r_2)/(s_1, s_2)) = (0, 0)$, then there are $u_i \in S_i$ such that $u_i r_i = 0$ for $i = 1, 2$. Hence $(u_1, u_2)(r_1, r_2) = (0, 0)$, so $(r_1, r_2)/(s_1, s_2) = 0$ in the source. Thus Φ is injective. □

Definition 2.4. A multiplicatively closed subset T of a ring R is said to be *saturated* if

$$ab \in T \iff a \in T \text{ and } b \in T.$$

Let T_1 and T_2 be saturated multiplicatively closed subsets of R_1 and R_2, respectively. Then $T_1 \times T_2$ is a saturated multiplicatively closed subset of $R_1 \times R_2$. There exists a unique smallest saturated multiplicatively closed subset $\overline{T}$ containing T, called the *saturation* of T. The saturation of $S_1 \times S_2$ is denoted by $\overline{S_1 \times S_2}$. In particular,

$$\overline{S_1 \times S_2} = \bigcap_{S_1 \times S_2 \subseteq T_1 \times T_2} (T_1 \times T_2),$$

where $T_1 \times T_2$ runs over all saturated multiplicatively closed subsets of $R_1 \times R_2$ containing $S_1 \times S_2$. Moreover, if $P_1 \in \operatorname{Spec}(R_1)$ and $P_2 \in \operatorname{Spec}(R_2)$, then $(R_1 - P_1) \times R_2$ and $R_1 \times (R_2 - P_2)$ are saturated subsets of $R_1 \times R_2$. Finally, for any multiplicatively closed subset S of a ring R, $\overline{S} = R - \bigcup_{P \cap S = \varnothing} P$, where $P \in \operatorname{Spec}(R)$.

Lemma 2.5. Let $S_1 \times S_2$ be a multiplicatively closed subset of $R_1 \times R_2$, let $P_1 \in \operatorname{Spec}(R_1)$, and let $P_2 \in \operatorname{Spec}(R_2)$. Then:

(1) $(P_1 \times R_2) \cap (S_1 \times S_2) = \varnothing$ if and only if $(P_1 \times R_2) \cap (\overline{S_1 \times S_2}) = \varnothing$.
(2) $(R_1 \times P_2) \cap (S_1 \times S_2) = \varnothing$ if and only if $(R_1 \times P_2) \cap (\overline{S_1 \times S_2}) = \varnothing$.

Proof . (1) If $(P_1 \times R_2) \cap (\overline{S_1 \times S_2}) = \varnothing$, then the equality $(P_1 \times R_2) \cap (S_1 \times S_2) = \varnothing$ follows immediately from $S_1 \times S_2 \subseteq \overline{S_1 \times S_2}$.

Conversely, assume that $(P_1 \times R_2) \cap (S_1 \times S_2) = \varnothing$. Then

$$S_1 \times S_2 \subseteq (R_1 \times R_2) - (P_1 \times R_2) = (R_1 - P_1) \times R_2.$$

Since $(R_1 - P_1) \times R_2$ is a saturated multiplicatively closed subset of $R_1 \times R_2$, we obtain

$$\overline{S_1 \times S_2} \subseteq (R_1 - P_1) \times R_2.$$

Hence

$$(P_1 \times R_2) \cap (\overline{S_1 \times S_2}) = \varnothing.$$

(2) The proof is entirely analogous to that of (1). □

Proposition 2.6. Let $S_1 \times S_2$ be a multiplicatively closed subset of $R_1 \times R_2$. Then

$$\overline{S_1 \times S_2} = \overline{S_1} \times \overline{S_2}$$
$$= (R_1 \times R_2) \setminus \left(\bigcup_{\substack{P_1 \in \mathrm{Spec}(R_1) \\ P_1 \cap S_1 = \emptyset}} (P_1 \times R_2) \ \cup \bigcup_{\substack{P_2 \in \mathrm{Spec}(R_2) \\ P_2 \cap S_2 = \emptyset}} (R_1 \times P_2) \right).$$

Proof . It is known that for any multiplicatively closed subset S of a ring R,

$$\overline{S} = R - \bigcup_{\substack{P \in \mathrm{Spec}(R) \\ P \cap S = \emptyset}} P.$$

Applying this formula to the product ring $R_1 \times R_2$, and using the description of prime ideals of a product ring,

$$\mathrm{Spec}(R_1 \times R_2) = \{P_1 \times R_2 : \ P_1 \in \mathrm{Spec}(R_1)\} \cup \{R_1 \times P_2 : \ P_2 \in \mathrm{Spec}(R_2)\},$$

we obtain

$$\overline{S_1 \times S_2} = (R_1 \times R_2) \setminus \left(\bigcup_{\substack{P_1 \in \mathrm{Spec}(R_1) \\ P_1 \cap S_1 = \emptyset}} (P_1 \times R_2) \ \cup \bigcup_{\substack{P_2 \in \mathrm{Spec}(R_2) \\ P_2 \cap S_2 = \emptyset}} (R_1 \times P_2) \right).$$

Taking complements and using the definitions of $\overline{S_1}$ and $\overline{S_2}$, this becomes

$$\overline{S_1 \times S_2} = \overline{S_1} \times \overline{S_2}.$$

This completes the proof. □

Proposition 2.7. Let $S_1 \times S_2$ be a multiplicatively closed subset of $R_1 \times R_2$, and let

$$\iota : R_1 \times R_2 \longrightarrow (S_1 \times S_2)^{-1}(R_1 \times R_2)$$

be the natural homomorphism defined by $\iota(r_1, r_2) = \frac{(r_1, r_2)}{(1,1)}$. If $I_1 \times I_2$ is an ideal of $R_1 \times R_2$, then its extension $(I_1 \times I_2)^e$ in $(S_1 \times S_2)^{-1}(R_1 \times R_2)$ is given by $(S_1 \times S_2)^{-1}(I_1 \times I_2)$, and

$$(I_1 \times I_2)^e = \Big\{ (\lambda_1, \lambda_2) \in (S_1 \times S_2)^{-1}(R_1 \times R_2) :$$
$$(\lambda_1, \lambda_2) = \frac{(a_1, a_2)}{(s_1, s_2)} \text{ for some } (a_1, a_2) \in I_1 \times I_2 \text{ and } (s_1, s_2) \in S_1 \times S_2 \Big\}.$$

Proof . For any ideal I of a ring R, its extension to $S^{-1}R$ is $S^{-1}I$; see [2, Proposition 3.11(i)] or [10, Lemma 10.9.16]. Applying this standard fact to $R_1 \times R_2$ and $I_1 \times I_2$ gives the asserted description. The displayed fraction description follows directly from the definition of $S^{-1}(I_1 \times I_2)$. □

Corollary 2.8. Let the situation be as in Proposition 2.7. Then the following conditions are equivalent:

(1) $(I_1 \times I_2)^e = (S_1 \times S_2)^{-1}(R_1 \times R_2)$;
(2) $(I_1 \times I_2) \cap (S_1 \times S_2) \neq \emptyset$.

Proof . (2) $\Longrightarrow$ (1) Suppose that $(I_1 \times I_2) \cap (S_1 \times S_2) \neq \emptyset$. Then there exists

$$(a_1, a_2) \in (I_1 \times I_2) \cap (S_1 \times S_2).$$

By Proposition 2.7, $\frac{(a_1,a_2)}{(a_1,a_2)} \in (I_1 \times I_2)^e$, and hence $\frac{(1,1)}{(1,1)} \in (I_1 \times I_2)^e$. Therefore

$$(I_1 \times I_2)^e = (S_1 \times S_2)^{-1}(R_1 \times R_2).$$

(1) $\Longrightarrow$ (2) Assume that $(I_1 \times I_2)^e = (S_1 \times S_2)^{-1}(R_1 \times R_2)$. Then $\frac{(1,1)}{(1,1)} \in (I_1 \times I_2)^e$. By Proposition 2.7, there exist $(a_1, a_2) \in I_1 \times I_2$ and $(s_1, s_2) \in S_1 \times S_2$ such that

$$\frac{(1,1)}{(1,1)} = \frac{(a_1, a_2)}{(s_1, s_2)}.$$

Thus there exists $(u_1, u_2) \in S_1 \times S_2$ such that

$$(u_1, u_2)\big((s_1, s_2) - (a_1, a_2)\big) = (0, 0).$$

This means that

$$u_1 s_1 = u_1 a_1, \qquad u_2 s_2 = u_2 a_2.$$

Since $a_1 \in I_1$ and $a_2 \in I_2$, we have $u_1 a_1 \in I_1$ and $u_2 a_2 \in I_2$. Also $u_1 s_1 \in S_1$ and $u_2 s_2 \in S_2$. Consequently,

$$(u_1 s_1, u_2 s_2) = (u_1 a_1, u_2 a_2) \in (I_1 \times I_2) \cap (S_1 \times S_2).$$

Hence $(I_1 \times I_2) \cap (S_1 \times S_2) \neq \emptyset$. □

Remark 2.9. Let $I_1 \times I_2$ and $J_1 \times J_2$ be ideals of $R_1 \times R_2$. Then the following properties hold:

(1) $(I_1 \times I_2)^e + (J_1 \times J_2)^e = \big[(I_1 + J_1) \times (I_2 + J_2)\big]^e$.
(2) $(I_1 \times I_2)^e (J_1 \times J_2)^e = \big[(I_1 J_1) \times (I_2 J_2)\big]^e$.
(3) $\big(\sqrt{I_1 \times I_2}\big)^e = \sqrt{(I_1 \times I_2)^e}$.

Proof . The proof is routine and is therefore omitted. □

Proposition 2.10. Let the situation be as in Proposition 2.7, and let $P_1 \times R_2$ and $R_1 \times P_2$ be prime ideals of $R_1 \times R_2$ such that

$$(P_1 \times R_2) \cap (S_1 \times S_2) = \varnothing, \qquad (R_1 \times P_2) \cap (S_1 \times S_2) = \varnothing.$$

Then $(P_1 \times R_2)^e$ and $(R_1 \times P_2)^e$ are prime ideals of $(S_1 \times S_2)^{-1}(R_1 \times R_2)$.

Proof . We prove the assertion for $(P_1 \times R_2)^e$; the proof for $(R_1 \times P_2)^e$ is similar. Let

$$\frac{(a_1, a_2)}{(s_1, s_2)} \cdot \frac{(b_1, b_2)}{(t_1, t_2)} \in (P_1 \times R_2)^e,$$

where

$$\frac{(a_1, a_2)}{(s_1, s_2)}, \frac{(b_1, b_2)}{(t_1, t_2)} \in (S_1 \times S_2)^{-1}(R_1 \times R_2).$$

Then

$$\frac{(a_1 b_1, a_2 b_2)}{(s_1 t_1, s_2 t_2)} \in (P_1 \times R_2)^e.$$

By Proposition 2.7, there exists $(u_1, u_2) \in S_1 \times S_2$ such that

$$(u_1, u_2)(a_1 b_1, a_2 b_2) \in P_1 \times R_2.$$

Since $(P_1 \times R_2) \cap (S_1 \times S_2) = \varnothing$, we have $u_1 \notin P_1$. Hence $a_1 b_1 \in P_1$. Since P_1 is a prime ideal of R_1, we obtain $a_1 \in P_1$ or $b_1 \in P_1$. Therefore

$$(a_1, a_2) \in P_1 \times R_2 \quad \text{or} \quad (b_1, b_2) \in P_1 \times R_2.$$

Consequently,

$$\frac{(a_1, a_2)}{(s_1, s_2)} \in (P_1 \times R_2)^e \quad \text{or} \quad \frac{(b_1, b_2)}{(t_1, t_2)} \in (P_1 \times R_2)^e.$$

Thus $(P_1 \times R_2)^e$ is a prime ideal of $(S_1 \times S_2)^{-1}(R_1 \times R_2)$. □

Proposition 2.11. If J is an ideal of $(S_1 \times S_2)^{-1}(R_1 \times R_2)$, then $J^{ce} = J$.

Proof . It is well known that $J^{ce} \subseteq J$. To prove the reverse inclusion, let $\frac{(a,b)}{(s,t)} \in J$. Since $\frac{(s,t)}{(1,1)} \in (S_1 \times S_2)^{-1}(R_1 \times R_2)$, we have

$$\frac{(a,b)}{(1,1)} = \frac{(a,b)}{(s,t)} \cdot \frac{(s,t)}{(1,1)} \in J.$$

Thus $\iota(a,b) \in J$, where

$$\iota : R_1 \times R_2 \longrightarrow (S_1 \times S_2)^{-1}(R_1 \times R_2)$$

is the natural homomorphism defined by $\iota(r_1, r_2) = \frac{(r_1,r_2)}{(1,1)}$. This means that

$$(a,b) \in J^c = \iota^{-1}(J).$$

Consequently,

$$\frac{(a,b)}{(s,t)} = \frac{(a,b)}{(1,1)} \cdot \frac{(1,1)}{(s,t)} = \iota(a,b)\frac{(1,1)}{(s,t)} \in J^{ce}.$$

This proves that $J \subseteq J^{ce}$, and hence $J^{ce} = J$. □

Remark 2.12. Let R_1 and R_2 be Noetherian (resp. Artinian) rings. Then

$$(S_1 \times S_2)^{-1}(R_1 \times R_2)$$

is a Noetherian (resp. Artinian) ring.

3 Results on $(S_1 \times S_2)^{-1}(M_1 \times M_2)$ as an $(S_1 \times S_2)^{-1}(R_1 \times R_2)$-module

The module $(S_1 \times S_2)^{-1}(M_1 \times M_2)$ is the standard localization of the $R_1 \times R_2$-module $M_1 \times M_2$. Its universal property, functoriality, and exactness are standard; see [10, Section 10.9] and [4, Section 6]. We record only the consequences needed for the product-specific results below.

Proposition 3.1. The set $(S_1 \times S_2)^{-1}(M_1 \times M_2)$ forms an $(S_1 \times S_2)^{-1}(R_1 \times R_2)$-module.

Proof . This is the standard module structure on a localization, applied to the ring $R_1 \times R_2$, its multiplicative subset $S_1 \times S_2$, and the module $M_1 \times M_2$; see [10, Definition 10.9.6]. □

Remark 3.2. Let

$$\iota : R_1 \times R_2 \longrightarrow (S_1 \times S_2)^{-1}(R_1 \times R_2), \qquad \iota(r,t) = \frac{(r,t)}{(1,1)}$$

be the natural ring homomorphism, so that $(S_1 \times S_2)^{-1}(R_1 \times R_2)$ is an $R_1 \times R_2$-module via ι. Similarly, let

$$f_1 : M_1 \longrightarrow S_1^{-1}M_1, \qquad f_1(m_1) = \frac{m_1}{1},$$

$$f_2 : M_2 \longrightarrow S_2^{-1}M_2, \qquad f_2(m_2) = \frac{m_2}{1},$$

be the natural R_1- and R_2-module homomorphisms, respectively. Define

$$f : M_1 \times M_2 \longrightarrow (S_1 \times S_2)^{-1}(M_1 \times M_2), \qquad f(m_1,m_2) = \frac{(m_1,m_2)}{(1,1)}.$$

Then f is an $R_1 \times R_2$-module homomorphism. If $\frac{(m_1,m_2)}{(s,t)} = 0$ in $(S_1 \times S_2)^{-1}(M_1 \times M_2)$, then $(u_1,u_2)(m_1,m_2) = (0,0)$ for some $(u_1,u_2) \in S_1 \times S_2$, so $u_1m_1 = 0$ and $u_2m_2 = 0$. Hence $\frac{m_1}{s} = 0$ and $\frac{m_2}{t} = 0$. Consequently,

$$\ker f = \left\{(m_1,m_2) \in M_1 \times M_2 : \frac{(m_1,m_2)}{(1,1)} = 0\right\} = \ker f_1 \times \ker f_2.$$

Proposition 3.3. Let M_1, N_1 be R_1-modules and M_2, N_2 be R_2-modules. Assume that

$$h : M_1 \times M_2 \longrightarrow N_1 \times N_2$$

is an $R_1 \times R_2$-module homomorphism. Then the mapping

$$h^* : (S_1 \times S_2)^{-1}(M_1 \times M_2) \longrightarrow (S_1 \times S_2)^{-1}(N_1 \times N_2)$$

defined by

$$h^*\left(\frac{(m_1,m_2)}{(s,t)}\right) = \frac{h(m_1,m_2)}{(s,t)}$$

is an $(S_1 \times S_2)^{-1}(R_1 \times R_2)$-module homomorphism.

Proof . The map is the standard map induced by a module homomorphism under localization, $m/s \mapsto h(m)/s$; its well-definedness and linearity follow from the defining equivalence relation. See [10, Section 10.9]. □

Remark 3.4. In Proposition 3.3, the mapping h^* is also denoted by $(S_1 \times S_2)^{-1}(h)$, and its kernel is

$$\begin{aligned}\ker(S_1 \times S_2)^{-1}(h) &= \left\{ \frac{(m_1, m_2)}{(s_1, s_2)} \in (S_1 \times S_2)^{-1}(M_1 \times M_2) : \frac{h(m_1, m_2)}{(s_1, s_2)} = 0 \right\} \\ &= \left\{ \frac{(m_1, m_2)}{(s_1, s_2)} : \exists (u_1, u_2) \in S_1 \times S_2 \text{ such that } (u_1, u_2)h(m_1, m_2) = (0, 0) \right\} \\ &= \left\{ \frac{(m_1, m_2)}{(s_1, s_2)} : \exists (u_1, u_2) \in S_1 \times S_2 \text{ such that } (u_1 m_1, u_2 m_2) \in \ker h \right\}.\end{aligned}$$

Proposition 3.5. Let M_1, N_1, P_1 be R_1-modules and M_2, N_2, P_2 be R_2-modules. Let

$$f : M_1 \times M_2 \longrightarrow N_1 \times N_2, \qquad g : N_1 \times N_2 \longrightarrow P_1 \times P_2$$

be two $R_1 \times R_2$-module homomorphisms. Then the composition

$$g \circ f : M_1 \times M_2 \longrightarrow P_1 \times P_2$$

is an $R_1 \times R_2$-module homomorphism, and

$$(S_1 \times S_2)^{-1}(g \circ f) = (S_1 \times S_2)^{-1}(g) \circ (S_1 \times S_2)^{-1}(f).$$

Proof . This is the functoriality of localization; see [10, Section 10.9]. □

Proposition 3.6. Let

$$0 \longrightarrow L_1 \times L_2 \xrightarrow{f} M_1 \times M_2 \xrightarrow{g} N_1 \times N_2 \longrightarrow 0$$

be an exact sequence of $R_1 \times R_2$-modules and homomorphisms. Then the sequence

$$\begin{aligned}0 \longrightarrow (S_1 \times S_2)^{-1}(L_1 \times L_2) &\xrightarrow{(S_1 \times S_2)^{-1}(f)} (S_1 \times S_2)^{-1}(M_1 \times M_2) \\ &\xrightarrow{(S_1 \times S_2)^{-1}(g)} (S_1 \times S_2)^{-1}(N_1 \times N_2) \longrightarrow 0\end{aligned}$$

is an exact sequence of $(S_1 \times S_2)^{-1}(R_1 \times R_2)$-modules and homomorphisms.

Proof . Localization is an exact functor on modules; see [10, Proposition 10.9.12] or [4, Theorem 6.2]. □

Remark 3.7. The exactness of the localization functor established in Proposition 3.6 has the following consequences:

(1) It preserves kernels, images, cokernels, and short exact sequences.
(2) For any submodule $N_1 \times N_2 \leq M_1 \times M_2$, the natural isomorphism

$$(S_1 \times S_2)^{-1}(M_1 \times M_2)/(S_1 \times S_2)^{-1}(N_1 \times N_2) \cong (S_1 \times S_2)^{-1}((M_1 \times M_2)/(N_1 \times N_2))$$

holds (see Proposition 3.18 below).
(3) The functor is additive, commutes with finite direct sums and finite products, and preserves finite limits and finite colimits.

Corollary 3.8. Let $M_1 \times M_2$ be an $R_1 \times R_2$-module and let $I_1 \times I_2$ be an ideal of $R_1 \times R_2$. Assume that

$$\iota : R_1 \times R_2 \longrightarrow (S_1 \times S_2)^{-1}(R_1 \times R_2), \qquad \iota(r_1, r_2) = \frac{(r_1, r_2)}{(1, 1)}$$

is the natural homomorphism. Then

$$(S_1 \times S_2)^{-1}\big((I_1 \times I_2)(M_1 \times M_2)\big) = (I_1 \times I_2)^e\,(S_1 \times S_2)^{-1}(M_1 \times M_2).$$

Corollary 3.9. Let $N_1 \times N_2$ and $L_1 \times L_2$ be two submodules of $M_1 \times M_2$. Then:

(1) $(S_1 \times S_2)^{-1}\big((L_1 \times L_2) + (N_1 \times N_2)\big) = (S_1 \times S_2)^{-1}(L_1 \times L_2) + (S_1 \times S_2)^{-1}(N_1 \times N_2)$.
(2) $(S_1 \times S_2)^{-1}\big((L_1 \times L_2) \cap (N_1 \times N_2)\big) = (S_1 \times S_2)^{-1}(L_1 \times L_2) \cap (S_1 \times S_2)^{-1}(N_1 \times N_2)$.

Proof . Both assertions follow from the exactness of localization and the fact that

$$(L_1 \times L_2) + (N_1 \times N_2) = (L_1 + N_1) \times (L_2 + N_2),$$
$$(L_1 \times L_2) \cap (N_1 \times N_2) = (L_1 \cap N_1) \times (L_2 \cap N_2),$$

together with the coordinatewise description of localization of product modules. □

In the following proposition, the structure of submodules of $(S_1 \times S_2)^{-1}(M_1 \times M_2)$ is described.

Proposition 3.10. Let T be a submodule of $(S_1 \times S_2)^{-1}(M_1 \times M_2)$. Then there are submodules $N_i \leq M_i$ such that $T = (S_1 \times S_2)^{-1}(N_1 \times N_2)$.

Proof . Let $N = T^c$ be the inverse image of T under the natural map $M_1 \times M_2 \to (S_1 \times S_2)^{-1}(M_1 \times M_2)$. The standard submodule correspondence for localization gives $T = (S_1 \times S_2)^{-1}N$; see [10, Lemma 10.9.15]. Moreover, every submodule of $M_1 \times M_2$ is of the form $N_1 \times N_2$: indeed, the idempotents $(1, 0)$ and $(0, 1)$ split each of its elements into its two coordinate components. Thus $N = N_1 \times N_2$ for suitable submodules $N_i \leq M_i$. □

Proposition 3.11. Let $S_1^{-1}M_1$ be an R_1-module and $S_2^{-1}M_2$ be an R_2-module. Assume that

$$f_1 : M_1 \longrightarrow S_1^{-1}M_1, \qquad f_1(m_1) = \frac{m_1}{1_{R_1}},$$

is an R_1-module homomorphism and

$$f_2 : M_2 \longrightarrow S_2^{-1}M_2, \qquad f_2(m_2) = \frac{m_2}{1_{R_2}},$$

is an R_2-module homomorphism. Then the mapping

$$f : M_1 \times M_2 \longrightarrow S_1^{-1}M_1 \times S_2^{-1}M_2, \qquad f(m_1, m_2) = (f_1(m_1), f_2(m_2)),$$

is an $R_1 \times R_2$-module homomorphism with $\ker f = \ker f_1 \times \ker f_2$.

Proof . The proof is routine and is therefore omitted. □

Remark 3.12. Let the situation be as in Proposition 3.11, and assume that $S_1 \times S_2$ acts regularly on $M_1 \times M_2$, that is, for every $(s_1, s_2) \in S_1 \times S_2$ and every $(m_1, m_2) \in M_1 \times M_2$,

$$(s_1, s_2)(m_1, m_2) = (0_{M_1}, 0_{M_2}) \quad \Longrightarrow \quad (m_1, m_2) = (0_{M_1}, 0_{M_2}).$$

Since $\ker f = (0_{M_1}, 0_{M_2})$, the $R_1 \times R_2$-module $M_1 \times M_2$ embeds into the $R_1 \times R_2$-module $S_1^{-1}M_1 \times S_2^{-1}M_2$.

Theorem 3.13 (Localization of a product module). Let M_1 be an R_1-module and M_2 an R_2-module. The mapping

$$\Psi : (S_1 \times S_2)^{-1}(M_1 \times M_2) \longrightarrow S_1^{-1}M_1 \times S_2^{-1}M_2$$

defined by

$$\Psi\left(\frac{(m_1, m_2)}{(s_1, s_2)}\right) = \left(\frac{m_1}{s_1}, \frac{m_2}{s_2}\right)$$

is an isomorphism of $(S_1 \times S_2)^{-1}(R_1 \times R_2)$-modules.

Proof . *Well-definedness.* Let $\frac{(m_1,m_2)}{(s_1,s_2)} = \frac{(m_1',m_2')}{(s_1',s_2')}$. Then

$$(u_1 s_1' m_1, u_2 s_2' m_2) = (u_1 s_1 m_1', u_2 s_2 m_2')$$

for some $(u_1, u_2) \in S_1 \times S_2$. Consequently,

$$u_1 s_1' m_1 = u_1 s_1 m_1', \qquad u_2 s_2' m_2 = u_2 s_2 m_2',$$

which imply

$$\frac{m_1}{s_1} = \frac{m_1'}{s_1'}, \qquad \frac{m_2}{s_2} = \frac{m_2'}{s_2'}.$$

Thus Ψ is well-defined.

Additivity and linearity. The computation

$$\begin{aligned}\Psi\left(\frac{(m_1, m_2)}{(s_1, s_2)} + \frac{(m_1', m_2')}{(s_1', s_2')}\right) &= \Psi\left(\frac{(s_1' m_1 + s_1 m_1',\ s_2' m_2 + s_2 m_2')}{(s_1 s_1',\ s_2 s_2')}\right) \\ &= \left(\frac{m_1}{s_1} + \frac{m_1'}{s_1'}, \frac{m_2}{s_2} + \frac{m_2'}{s_2'}\right)\end{aligned}$$

shows that Ψ preserves addition; linearity over $(S_1 \times S_2)^{-1}(R_1 \times R_2)$ follows similarly.

Surjectivity. For any $(m_1/s_1, m_2/s_2) \in S_1^{-1}M_1 \times S_2^{-1}M_2$, we have

$$(m_1/s_1, m_2/s_2) = \Psi\big((m_1, m_2)/(s_1, s_2)\big).$$

Injectivity. Assume $\Psi((m_1, m_2)/(s_1, s_2)) = (0, 0)$. Then $m_1/s_1 = 0$ in $S_1^{-1}M_1$ and $m_2/s_2 = 0$ in $S_2^{-1}M_2$, so there exist $u_1 \in S_1$ and $u_2 \in S_2$ with $u_1 m_1 = 0$ and $u_2 m_2 = 0$. Hence $(u_1, u_2)(m_1, m_2) = (0, 0)$, and therefore $(m_1, m_2)/(s_1, s_2) = 0$ in the localization. Thus Ψ is injective, and consequently an isomorphism. □

Remark 3.14. As part of the proof of Theorem 3.13, the kernel of the natural $R_1 \times R_2$-module homomorphism

$$M_1 \times M_2 \longrightarrow S_1^{-1}M_1 \times S_2^{-1}M_2, \qquad (m_1, m_2) \longmapsto (m_1/1, m_2/1),$$

is $\ker f_1 \times \ker f_2$, where $f_i \colon M_i \to S_i^{-1}M_i$ are the natural maps. Equivalently, it is precisely the set of $S_1 \times S_2$-torsion elements of $M_1 \times M_2$. In particular, if $S_1 \times S_2$ acts regularly on $M_1 \times M_2$, then $M_1 \times M_2$ embeds into $S_1^{-1}M_1 \times S_2^{-1}M_2$.

Remark 3.15. We shall need the structure of submodules of the $R_1 \times R_2$-module $M_1 \times M_2$. Let N be a submodule of $M_1 \times M_2$, and consider

$$\rho_1 : M_1 \times M_2 \longrightarrow M_1 \times 0, \qquad \rho_1(m_1, m_2) = (m_1, 0),$$

$$\rho_2 : M_1 \times M_2 \longrightarrow 0 \times M_2, \qquad \rho_2(m_1, m_2) = (0, m_2).$$

Clearly, ρ_1 and ρ_2 are $R_1 \times R_2$-module homomorphisms. Define

$$N_1 = \{n_1 \in M_1 \mid (n_1, 0) \in \rho_1(N)\}, \qquad N_2 = \{n_2 \in M_2 \mid (0, n_2) \in \rho_2(N)\}.$$

Since $\rho_1(N)$ and $\rho_2(N)$ are submodules of $M_1 \times 0$ and $0 \times M_2$, respectively, N_1 and N_2 are submodules of M_1 and M_2, respectively. We claim that $N = N_1 \times N_2$.

If $(n_1, n_2) \in N$, then

$$\rho_1(n_1, n_2) = (n_1, 0), \qquad \rho_2(n_1, n_2) = (0, n_2),$$

so $n_1 \in N_1$ and $n_2 \in N_2$. Hence $(n_1, n_2) \in N_1 \times N_2$.

Conversely, let $(n_1, n_2) \in N_1 \times N_2$. Then

$$\rho_1(n_1, n_2') = (n_1, 0), \qquad \rho_2(n_1', n_2) = (0, n_2)$$

for some $(n_1, n_2'), (n_1', n_2) \in N$. Since N is a submodule of $M_1 \times M_2$, we have

$$(1_{R_1}, 0)(n_1, n_2') \in N, \qquad (0, 1_{R_2})(n_1', n_2) \in N,$$

which imply $(n_1, 0) \in N$ and $(0, n_2) \in N$. Therefore,

$$(n_1, n_2) = (n_1, 0) + (0, n_2) \in N,$$

so $N_1 \times N_2 \subseteq N$. Thus $N = N_1 \times N_2$ for some submodules $N_1 \leq M_1$ and $N_2 \leq M_2$.

Corollary 3.16. Let T be a submodule of the $S_1^{-1}R_1 \times S_2^{-1}R_2$-module $S_1^{-1}M_1 \times S_2^{-1}M_2$. Then $T = S_1^{-1}N_1 \times S_2^{-1}N_2$ for some submodule N_1 of M_1 and some submodule N_2 of M_2.

Proof . By Remark 3.15, every submodule $T \leq S_1^{-1}M_1 \times S_2^{-1}M_2$ is of the form $T = T_1 \times T_2$, where $T_1 \leq S_1^{-1}M_1$ and $T_2 \leq S_2^{-1}M_2$. For $i = 1, 2$, define

$$N_i = \{m_i \in M_i : \tfrac{m_i}{1_{R_i}} \in T_i\}.$$

It is routine to show that $T_i = S_i^{-1}N_i$ for $i = 1, 2$. Therefore, $T = S_1^{-1}N_1 \times S_2^{-1}N_2$, as required. □

In the following, we review the $R_1 \times R_2$-module $\frac{M_1 \times M_2}{N_1 \times N_2}$, which is called the quotient of $M_1 \times M_2$ by $N_1 \times N_2$.

Remark 3.17. Let $M_1 \times M_2$ be an $R_1 \times R_2$-module and $N_1 \times N_2 \leq M_1 \times M_2$ a submodule. The abelian group $\frac{M_1 \times M_2}{N_1 \times N_2}$ inherits an $R_1 \times R_2$-module structure from $M_1 \times M_2$, where

$$\frac{M_1 \times M_2}{N_1 \times N_2} = \{(m_1, m_2) + N_1 \times N_2 \mid (m_1, m_2) \in M_1 \times M_2\}.$$

Scalar multiplication is defined in the natural way:

$$(r_1, r_2) \cdot \big((m_1, m_2) + N_1 \times N_2\big) = (r_1 m_1, r_2 m_2) + N_1 \times N_2.$$

Then $\frac{M_1 \times M_2}{N_1 \times N_2}$ is an $R_1 \times R_2$-module. Moreover, the mapping

$$\psi : \frac{M_1 \times M_2}{N_1 \times N_2} \longrightarrow \frac{M_1}{N_1} \times \frac{M_2}{N_2}, \quad \psi\big((m_1, m_2) + N_1 \times N_2\big) = (m_1 + N_1, m_2 + N_2),$$

is an isomorphism, and hence

$$\frac{M_1 \times M_2}{N_1 \times N_2} \cong \frac{M_1}{N_1} \times \frac{M_2}{N_2}.$$

Now endow

$$(S_1 \times S_2)^{-1}\left(\frac{M_1 \times M_2}{N_1 \times N_2}\right) = \left\{\frac{(m_1, m_2) + N_1 \times N_2}{(s_1, s_2)} : \frac{(m_1, m_2)}{(s_1, s_2)} \in (S_1 \times S_2)^{-1}(M_1 \times M_2)\right\}$$

with an $(S_1 \times S_2)^{-1}(R_1 \times R_2)$-module structure by

$$\begin{aligned}\frac{(m_1, m_2) + N_1 \times N_2}{(s_1, s_2)} + \frac{(m_1', m_2') + N_1 \times N_2}{(s_1', s_2')} \\ = \frac{(s_1' m_1 + s_1 m_1',\ s_2' m_2 + s_2 m_2') + N_1 \times N_2}{(s_1 s_1',\ s_2 s_2')},\end{aligned}$$

and

$$\frac{(r_1, r_2)}{(s_1', s_2')} \cdot \frac{(m_1, m_2) + N_1 \times N_2}{(s_1, s_2)} = \frac{(r_1 m_1, r_2 m_2) + N_1 \times N_2}{(s_1' s_1,\ s_2' s_2)}.$$

Proposition 3.18. Let $N_1 \times N_2$ be a submodule of $M_1 \times M_2$. Then

$$(S_1 \times S_2)^{-1}\left(\frac{M_1 \times M_2}{N_1 \times N_2}\right) \cong \frac{(S_1 \times S_2)^{-1}(M_1 \times M_2)}{(S_1 \times S_2)^{-1}(N_1 \times N_2)}.$$

Proof . This is the standard compatibility of localization with quotient modules; see [10, Lemma 10.9.13]. Applying it to $R_1 \times R_2$, $S_1 \times S_2$, and $N_1 \times N_2 \leq M_1 \times M_2$ yields the stated isomorphism. □

Corollary 3.19. Let $N_1 \times N_2$ be a submodule of $M_1 \times M_2$. Then

$$(S_1 \times S_2)^{-1}\left(\frac{M_1 \times M_2}{N_1 \times N_2}\right) \cong \frac{S_1^{-1}M_1 \times S_2^{-1}M_2}{S_1^{-1}N_1 \times S_2^{-1}N_2}.$$

Proof . Combine Proposition 3.18 with Theorem 3.13 (applied to both $M_1 \times M_2$ and $N_1 \times N_2$). □

Proposition 3.20. Let N_1, L_1 be submodules of M_1 and N_2, L_2 be submodules of M_2. Then

$$(L_1 \times L_2 :_{R_1 \times R_2} N_1 \times N_2)^e \subseteq \big((S_1 \times S_2)^{-1}(L_1 \times L_2) :_{(S_1 \times S_2)^{-1}(R_1 \times R_2)} (S_1 \times S_2)^{-1}(N_1 \times N_2)\big).$$

Proof . Suppose that $\lambda \in (L_1 \times L_2 :_{R_1 \times R_2} N_1 \times N_2)^e$, so $\lambda = \frac{(r_1, r_2)}{(s_1, s_2)}$ with

$$(r_1, r_2) \in (L_1 \times L_2 :_{R_1 \times R_2} N_1 \times N_2).$$

Then $(r_1, r_2)(N_1 \times N_2) \subseteq L_1 \times L_2$. Now, let $\frac{(n_1, n_2)}{(s_1', s_2')} \in (S_1 \times S_2)^{-1}(N_1 \times N_2)$. Since $r_1 n_1 \in L_1$ and $r_2 n_2 \in L_2$, we have $(r_1 n_1, r_2 n_2) \in L_1 \times L_2$, and hence

$$\frac{(r_1 n_1, r_2 n_2)}{(s_1 s_1', s_2 s_2')} \in (S_1 \times S_2)^{-1}(L_1 \times L_2).$$

Therefore,

$$\frac{(r_1, r_2)}{(s_1, s_2)} \cdot \frac{(n_1, n_2)}{(s_1', s_2')} \in (S_1 \times S_2)^{-1}(L_1 \times L_2),$$

and so

$$\lambda(S_1 \times S_2)^{-1}(N_1 \times N_2) \subseteq (S_1 \times S_2)^{-1}(L_1 \times L_2).$$

Equivalently,

$$\lambda \in \big((S_1 \times S_2)^{-1}(L_1 \times L_2) :_{(S_1 \times S_2)^{-1}(R_1 \times R_2)} (S_1 \times S_2)^{-1}(N_1 \times N_2)\big).$$

□

Corollary 3.21. Assume that $L_1 \times L_2 = 0$ in Proposition 3.20. Then

$$\mathrm{Ann}_{R_1\times R_2}(N_1 \times N_2)^e \subseteq \mathrm{Ann}_{(S_1\times S_2)^{-1}(R_1\times R_2)} \left((S_1 \times S_2)^{-1}(N_1 \times N_2)\right).$$

Corollary 3.22. Let the assumptions be as in Proposition 3.20, and let $N_1\times N_2$ be a finitely generated submodule of $M_1 \times M_2$. Then

$$(L_1\times L_2 :_{R_1\times R_2} N_1\times N_2)^e = \left((S_1\times S_2)^{-1}(L_1\times L_2) :_{(S_1\times S_2)^{-1}(R_1\times R_2)} (S_1\times S_2)^{-1}(N_1\times N_2)\right).$$

Proof . By Proposition 3.20, it is enough to prove that

$$\left((S_1\times S_2)^{-1}(L_1\times L_2) :_{(S_1\times S_2)^{-1}(R_1\times R_2)} (S_1\times S_2)^{-1}(N_1\times N_2)\right) \subseteq (L_1\times L_2 :_{R_1\times R_2} N_1\times N_2)^e.$$

Let

$$\lambda \in \left((S_1 \times S_2)^{-1}(L_1 \times L_2) :_{(S_1\times S_2)^{-1}(R_1\times R_2)} (S_1 \times S_2)^{-1}(N_1 \times N_2)\right),$$

so $\lambda = \frac{(r_1,r_2)}{(s_1,s_2)}$ with $(r_1, r_2) \in R_1 \times R_2$ and $(s_1, s_2) \in S_1 \times S_2$. Since $N_1 \times N_2$ is finitely generated, each of N_1 and N_2 is finitely generated. Write

$$N_1 \times N_2 = \langle (x_1, x_1'), \ldots, (x_n, x_n') \rangle$$

where $(x_i, x_i') \in N_1 \times N_2$ for $1 \leq i \leq n$. Since

$$\lambda(S_1 \times S_2)^{-1}(N_1 \times N_2) \subseteq (S_1 \times S_2)^{-1}(L_1 \times L_2),$$

we have

$$\lambda \frac{(x_i, x_i')}{(1, 1)} \in (S_1 \times S_2)^{-1}(L_1 \times L_2)$$

for every $i = 1, \ldots, n$. Now, for every i, write

$$\frac{(r_1x_i, r_2x_i')}{(s_1, s_2)} = \frac{(l_i, l_i')}{(t_i, t_i')}$$

with $(l_i, l_i') \in L_1 \times L_2$ and $(t_i, t_i') \in S_1 \times S_2$. Then

$$(u_i, u_i')(t_i, t_i')(r_1x_i, r_2x_i') = (u_i, u_i')(s_1, s_2)(l_i, l_i')$$

for some $(u_i, u_i') \in S_1 \times S_2$, so

$$(u_it_ir_1x_i, u_i't_i'r_2x_i') = (u_is_1l_i, u_i's_2l_i') \in L_1 \times L_2.$$

Let

$$s = u_1 t_1 \cdots u_n t_n \in S_1, \qquad s' = u_1' t_1' \cdots u_n' t_n' \in S_2.$$

Since S_1 and S_2 are multiplicatively closed, the finite products s and s' indeed belong to S_1 and S_2, respectively. Then

$$(s, s')(r_1, r_2)(x_i, x_i') \in L_1 \times L_2$$

for every $i = 1, \ldots, n$. Hence

$$(s, s')(r_1, r_2) \in (L_1 \times L_2 :_{R_1 \times R_2} N_1 \times N_2).$$

Finally,

$$\lambda = \frac{(r_1, r_2)}{(s_1, s_2)} = \frac{(s, s')(r_1, r_2)}{(s, s')(s_1, s_2)} \in (L_1 \times L_2 :_{R_1 \times R_2} N_1 \times N_2)^e.$$

This completes the proof. □

Remark 3.23 (Necessity of the finite generation hypothesis). The finite generation hypothesis in Corollary 3.22 is essential and cannot be removed. We illustrate this with an explicit example.

Let

$$R = \mathbb{Z}, \qquad S = \mathbb{Z} \setminus \{0\}, \qquad M = \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}, \qquad L = 0.$$

The module M is not finitely generated. We have $(0 :_R M) = 0$, because every nonzero integer n fails to annihilate infinitely many components $\mathbb{Z}/p\mathbb{Z}$ with $p \nmid n$. Hence $(0 :_R M)^e = 0$ in $\mathbb{Q}$. On the other hand, $M_S = M \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, since every component $\mathbb{Z}/p\mathbb{Z}$ is torsion and hence vanishes upon tensoring with $\mathbb{Q}$. Therefore, $(0_S :_{\mathbb{Q}} M_S) = (0 :_{\mathbb{Q}} 0) = \mathbb{Q}$. Consequently, $(0 :_R M)^e = 0 \neq \mathbb{Q} = (0_S :_{\mathbb{Q}} M_S)$, which shows that the equality of Corollary 3.22 fails without the finite generation hypothesis.

Remark 3.24. Suppose that $L_1 \times L_2 = 0$ in Corollary 3.22. Then

$$\operatorname{Ann}_{R_1 \times R_2}(N_1 \times N_2)^e = \operatorname{Ann}_{(S_1 \times S_2)^{-1}(R_1 \times R_2)} \left((S_1 \times S_2)^{-1}(N_1 \times N_2)\right).$$

Definition 3.25. An R-module M is called a *multiplication R-module* if for every submodule $N \leq M$ there exists an ideal $I \leq R$ such that $N = IM$. One easily verifies that if N is a submodule of a multiplication module M, then $N = (N :_R M)M$.

Suppose that N and K are submodules of a multiplication R-module M, with $N = IM$ and $K = JM$ for some ideals $I, J \leq R$. Then the product of N and K, denoted by NK, is defined by $NK = IJM$. We recall that the power N^m is defined by $N^m = (N :_R M)^m M$ (see [3, 6]).

Following the above definition, we discuss the Cartesian product of multiplication modules.

Remark 3.26. Let $N_1 \times N_2$ be a submodule of the $R_1 \times R_2$-module $M_1 \times M_2$. We have

$$(N_1 \times N_2 :_{R_1 \times R_2} M_1 \times M_2) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2).$$

Now, assume that M_1 is a multiplication R_1-module and M_2 is a multiplication R_2-module. Then

$$N_1 = (N_1 :_{R_1} M_1)M_1, \qquad N_2 = (N_2 :_{R_2} M_2)M_2,$$

so

$$N_1 \times N_2 = \big((N_1 :_{R_1} M_1)M_1\big) \times \big((N_2 :_{R_2} M_2)M_2\big).$$

We claim that

$$\big((N_1 :_{R_1} M_1)M_1\big) \times \big((N_2 :_{R_2} M_2)M_2\big) = \big((N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)\big)(M_1 \times M_2).$$

Indeed, let

$$(x, y) \in \big((N_1 :_{R_1} M_1)M_1\big) \times \big((N_2 :_{R_2} M_2)M_2\big).$$

Then $x \in (N_1 :_{R_1} M_1)M_1$ and $y \in (N_2 :_{R_2} M_2)M_2$. Without loss of generality, we may write

$$x = \sum_{i=1}^{n} r_i m_i, \qquad y = \sum_{i=1}^{n} r_i' m_i',$$

where $r_i \in (N_1 :_{R_1} M_1)$, $r_i' \in (N_2 :_{R_2} M_2)$, $m_i \in M_1$, and $m_i' \in M_2$. Then

$$(x, y) = \sum_{i=1}^{n} (r_i, r_i')(m_i, m_i'),$$

where

$$(r_i, r_i') \in (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2), \qquad (m_i, m_i') \in M_1 \times M_2.$$

Hence

$$(x, y) \in \big((N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)\big)(M_1 \times M_2).$$

The reverse inclusion is immediate. Consequently,

$$N_1 \times N_2 = (N_1 \times N_2 :_{R_1 \times R_2} M_1 \times M_2)(M_1 \times M_2).$$

This shows that $M_1 \times M_2$ is a multiplication $R_1 \times R_2$-module.

Proposition 3.27. Let M_1 be a multiplication R_1-module and M_2 a multiplication R_2-module. Then $(S_1 \times S_2)^{-1}(M_1 \times M_2)$ is a multiplication $(S_1 \times S_2)^{-1}(R_1 \times R_2)$-module.

Proof . Let $T \leq (S_1 \times S_2)^{-1}(M_1 \times M_2)$ be a submodule. By Proposition 3.10, there exists a submodule $N_1 \times N_2 \leq M_1 \times M_2$ such that $T = (S_1 \times S_2)^{-1}(N_1 \times N_2)$. Since M_i is a multiplication R_i-module, we have $N_i = (N_i :_{R_i} M_i)M_i$ for $i = 1, 2$. Set

$$I_i = (N_i :_{R_i} M_i), \qquad I_1 \times I_2 = I.$$

Then

$$N_1 \times N_2 = (I_1 \times I_2)(M_1 \times M_2) = I(M_1 \times M_2),$$

where in the last step we used Remark 3.26. Applying the exact functor $(S_1 \times S_2)^{-1}(-)$ and using Corollary 3.8, we get

$$\begin{aligned} T &= (S_1 \times S_2)^{-1}(N_1 \times N_2) \\ &= (S_1 \times S_2)^{-1}\big(I(M_1 \times M_2)\big) \\ &= I^e\,(S_1 \times S_2)^{-1}(M_1 \times M_2). \end{aligned}$$

Thus T is the product of an ideal of $(S_1 \times S_2)^{-1}(R_1 \times R_2)$ with the whole localized module, which proves that $(S_1 \times S_2)^{-1}(M_1 \times M_2)$ is a multiplication module. □

4 Transfer of ϕ-primality under localization

D. D. Anderson and M. Bataineh introduced generalizations of prime ideals in [1], where they defined the concept of a ϕ-prime ideal. Let $\mathcal{I}(R)$ denote the set of ideals of a ring R, and let $\phi : \mathcal{I}(R) \longrightarrow \mathcal{I}(R) \cup \{\varnothing\}$ be a function satisfying $\phi(I) \subseteq I$ for every nonempty $\phi(I)$. A proper ideal I of R is called a *ϕ-prime ideal* if for $a, b \in R$,

$$ab \in I \setminus \phi(I) \quad \Longrightarrow \quad a \in I \text{ or } b \in I.$$

Subsequently, Zamani [11] extended this notion to ϕ-prime submodules. Let $\mathcal{S}(M)$ be the set of all submodules of an R-module M, and let $\phi : \mathcal{S}(M) \longrightarrow \mathcal{S}(M) \cup \{\varnothing\}$ be a function satisfying $\phi(N) \subseteq N$ for every nonempty $\phi(N)$. A proper submodule N of M is called a *ϕ-prime submodule* if for $r \in R$ and $m \in M$,

$$rm \in N \setminus \phi(N) \quad \Longrightarrow \quad m \in N \text{ or } r \in (N :_R M).$$

Several properties of ϕ-prime ideals and ϕ-prime submodules were investigated in [1] and [11].

We state the following lemma, which will be needed in the sequel.

Lemma 4.1 (Classification of ϕ-prime ideals). Let

$$\phi_i : \mathcal{I}(R_i) \longrightarrow \mathcal{I}(R_i) \cup \{\varnothing\}, \qquad i = 1, 2,$$

and let

$$\phi : \mathcal{I}(R_1 \times R_2) \longrightarrow \mathcal{I}(R_1 \times R_2) \cup \{\varnothing\}$$

be the function defined by $\phi = \phi_1 \times \phi_2$. Then the ϕ-prime ideals of $R_1 \times R_2$ are of exactly one of the following three types:

(1) $I_1 \times I_2$, where I_i is a proper ideal of R_i with $\phi_i(I_i) = I_i$ for $i = 1, 2$;
(2) $I_1 \times R_2$, where I_1 is a ϕ_1-prime ideal of R_1 which must be prime if $\phi_2(R_2) \neq R_2$;
(3) $R_1 \times I_2$, where I_2 is a ϕ_2-prime ideal of R_2 which must be prime if $\phi_1(R_1) \neq R_1$.

Proof . See [1, Theorem 16]. □

Let

$$\phi_i : \mathcal{I}(R_i) \longrightarrow \mathcal{I}(R_i) \cup \{\varnothing\}, \qquad i = 1, 2,$$

and define $\phi = \phi_1 \times \phi_2$ coordinatewise on ideals of $R_1 \times R_2$, with the convention that $\phi(I_1 \times I_2) = \varnothing$ if either factor is empty. We assume throughout that $\phi(I) \subseteq I$ for every nonempty $\phi(I)$. For an ideal J of the localized ring, write

$$J^c = \{(r_1, r_2) \in R_1 \times R_2 : (r_1, r_2)/(1, 1) \in J\}$$

for its contraction. Every ideal of $R_1 \times R_2$ is a product ideal, and extension–contraction gives $J = (J^c)^e$. We therefore define the induced operation by

$$\phi_{S_1 \times S_2}(J) = (S_1 \times S_2)^{-1}\big(\phi(J^c)\big),$$

where the localization of the empty set is understood to be empty. Since $\phi(J^c) \subseteq J^c$, we have

$$\phi_{S_1 \times S_2}(J) \subseteq (S_1 \times S_2)^{-1}(J^c) = J.$$

This definition is independent of a presentation of J, since J^c is uniquely determined by J.

We say that a proper ideal $(I_1 \times I_2)^e$ of $(S_1 \times S_2)^{-1}(R_1 \times R_2)$ is a $\phi_{S_1 \times S_2}$-*prime ideal* if whenever

$$\frac{(x_1, y_1)}{(s_1, t_1)}, \frac{(x_2, y_2)}{(s_2, t_2)} \in (S_1 \times S_2)^{-1}(R_1 \times R_2)$$

satisfy

$$\frac{(x_1, y_1)}{(s_1, t_1)} \frac{(x_2, y_2)}{(s_2, t_2)} \in (I_1 \times I_2)^e \setminus \phi_{S_1 \times S_2}\big((I_1 \times I_2)^e\big),$$

it follows that

$$\frac{(x_1, y_1)}{(s_1, t_1)} \in (I_1 \times I_2)^e \quad \text{or} \quad \frac{(x_2, y_2)}{(s_2, t_2)} \in (I_1 \times I_2)^e.$$

By Lemma 4.1, the ϕ-prime ideals of $R_1 \times R_2$ fall into three types. If $I_1 \times I_2$ is a ϕ-prime ideal of Type (1) with $\phi_i(I_i) = I_i$, then

$$\phi_{S_1 \times S_2}\big((I_1 \times I_2)^e\big) = (S_1 \times S_2)^{-1}(I_1 \times I_2) = (I_1 \times I_2)^e,$$

and hence the set $(I_1 \times I_2)^e \setminus \phi_{S_1 \times S_2}((I_1 \times I_2)^e)$ is empty. Thus, in what follows, we examine the remaining two types of ϕ-prime ideals from Lemma 4.1.

Proposition 4.2. Let $\phi = \phi_1 \times \phi_2$ be the coordinatewise operation defined above, and let $\phi_{S_1\times S_2}$ be its induced operation via contraction. Let $I_1 \times R_2$ be a ϕ-prime ideal of $R_1 \times R_2$. Suppose that $S_1 \times S_2$ is a multiplicatively closed subset of $R_1 \times R_2$ such that

$$(I_1 \times R_2) \cap (S_1 \times S_2) = \varnothing$$

and

$$(S_1 \times S_2)^{-1}(\phi_1(I_1) \times R_2) \subseteq \phi_{S_1\times S_2}((I_1 \times R_2)^e).$$

Then $(I_1 \times R_2)^e$ is a $\phi_{S_1\times S_2}$-prime ideal of $(S_1 \times S_2)^{-1}(R_1 \times R_2)$.

Proof . Let

$$\frac{(x_1, y_1)}{(s_1, t_1)}, \frac{(x_2, y_2)}{(s_2, t_2)} \in (S_1 \times S_2)^{-1}(R_1 \times R_2)$$

such that

$$\frac{(x_1, y_1)}{(s_1, t_1)} \frac{(x_2, y_2)}{(s_2, t_2)} \in (I_1 \times R_2)^e \setminus \phi_{S_1\times S_2}((I_1 \times R_2)^e).$$

Then

$$\frac{(x_1x_2, y_1y_2)}{(s_1s_2, t_1t_2)} \in (I_1 \times R_2)^e$$

and

$$\frac{(x_1x_2, y_1y_2)}{(s_1s_2, t_1t_2)} \notin \phi_{S_1\times S_2}((I_1 \times R_2)^e).$$

Hence, there exist $(z_1, z_2) \in I_1 \times R_2$ and $(s, s') \in S_1 \times S_2$ such that

$$\frac{(x_1x_2, y_1y_2)}{(s_1s_2, t_1t_2)} = \frac{(z_1, z_2)}{(s, s')},$$

and moreover

$$\frac{(x_1x_2, y_1y_2)}{(s_1s_2, t_1t_2)} \notin (S_1 \times S_2)^{-1}(\phi_1(I_1) \times R_2).$$

It follows that there exists $(u_1, u_2) \in S_1 \times S_2$ such that

$$u_1sx_1x_2 = u_1s_1s_2z_1 \in I_1, \qquad u_2s'y_1y_2 = u_2t_1t_2z_2 \in R_2,$$

with $wx_1x_2 \notin \phi_1(I_1)$ for all $w \in S_1$. Setting $w = u_1 s \in S_1$ and $w' = u_2 s' \in S_2$, we obtain

$$wx_1x_2 \in I_1 \setminus \phi_1(I_1), \qquad w'y_1y_2 \in R_2.$$

Hence

$$(wx_1, w'y_1)(x_2, y_2) = (wx_1x_2, w'y_1y_2) \in (I_1 \times R_2) \setminus (\phi_1(I_1) \times R_2).$$

Since $I_1 \times R_2$ is a ϕ-prime ideal of $R_1 \times R_2$, we obtain

$$(wx_1, w'y_1) \in I_1 \times R_2 \quad \text{or} \quad (x_2, y_2) \in I_1 \times R_2.$$

Consequently,

$$\frac{(x_1, y_1)}{(s_1, t_1)} \in (I_1 \times R_2)^e \quad \text{or} \quad \frac{(x_2, y_2)}{(s_2, t_2)} \in (I_1 \times R_2)^e,$$

which completes the proof. □

Corollary 4.3. Let $\phi = \phi_1 \times \phi_2$ be the coordinatewise operation defined above, and let $\phi_{S_1 \times S_2}$ be its induced operation via contraction. Let $R_1 \times I_2$ be a ϕ-prime ideal of $R_1 \times R_2$. Suppose that $S_1 \times S_2$ is a multiplicatively closed subset of $R_1 \times R_2$ with

$$(R_1 \times I_2) \cap (S_1 \times S_2) = \varnothing$$

and

$$(S_1 \times S_2)^{-1}(R_1 \times \phi_2(I_2)) \subseteq \phi_{S_1 \times S_2}((R_1 \times I_2)^e).$$

Then $(R_1 \times I_2)^e$ is a $\phi_{S_1 \times S_2}$-prime ideal of $(S_1 \times S_2)^{-1}(R_1 \times R_2)$.

Proof . The proof is completely analogous to that of Proposition 4.2. □

Next, we study $\phi_{S_1 \times S_2}$-prime submodules of $(S_1 \times S_2)^{-1}(M_1 \times M_2)$. We begin with the saturation and contraction constructions, which are needed to make the definition of $\phi_{S_1 \times S_2}$ well-defined.

Definition 4.4 (Saturation of a submodule). Let $N \leq M$ be a submodule of an R-module M, and let $S \subseteq R$ be a multiplicatively closed subset. The *saturation of N with respect to S* is the submodule

$$N(S) = \{m \in M : \exists s \in S,\ sm \in N\}.$$

Clearly, $N \subseteq N(S) \leq M$, and $N(S)$ is the largest submodule of M whose localization at S equals $S^{-1}N$.

For a submodule $N_1 \times N_2 \leq M_1 \times M_2$, we have

$$(N_1 \times N_2)(S_1 \times S_2) = N_1(S_1) \times N_2(S_2).$$

It is easy to see that

$$S_1^{-1}(N_1(S_1)) = S_1^{-1}N_1, \qquad S_2^{-1}(N_2(S_2)) = S_2^{-1}N_2,$$

and

$$(S_1 \times S_2)^{-1}((N_1 \times N_2)(S_1 \times S_2)) = (S_1 \times S_2)^{-1}(N_1 \times N_2).$$

Definition 4.5 (Contraction of a submodule). Let $T \leq (S_1 \times S_2)^{-1}(M_1 \times M_2)$ be a submodule. The *contraction of* T is the submodule

$$T^c = \left\{(m_1, m_2) \in M_1 \times M_2 : \frac{(m_1, m_2)}{(1, 1)} \in T\right\}.$$

Lemma 4.6. Let $T \leq (S_1 \times S_2)^{-1}(M_1 \times M_2)$ be a submodule. Then:

(1) T^c is a submodule of $M_1 \times M_2$;
(2) $T = (S_1 \times S_2)^{-1}(T^c)$;
(3) $(T^c)(S_1 \times S_2) = T^c$.

Proof . (1) If $(m_1, m_2), (m_1', m_2') \in T^c$, then both fractions $\frac{(m_1,m_2)}{(1,1)}$ and $\frac{(m_1',m_2')}{(1,1)}$ belong to T. Since T is a submodule, we have

$$\frac{(m_1, m_2)}{(1, 1)} + \frac{(m_1', m_2')}{(1, 1)} = \frac{(m_1 + m_1', m_2 + m_2')}{(1, 1)} \in T,$$

and for $(r_1, r_2) \in R_1 \times R_2$,

$$(r_1, r_2)\frac{(m_1, m_2)}{(1, 1)} = \frac{(r_1m_1, r_2m_2)}{(1, 1)} \in T.$$

Hence T^c is a submodule of $M_1 \times M_2$.

(2) Let $(m_1, m_2) \in T^c$ and $(s_1, s_2) \in S_1 \times S_2$. Then

$$\frac{(m_1, m_2)}{(s_1, s_2)} = \frac{(m_1, m_2)}{(1, 1)} \cdot \frac{(1, 1)}{(s_1, s_2)} \in (S_1 \times S_2)^{-1}(T^c).$$

Conversely, let $\lambda = \frac{(m_1,m_2)}{(s_1,s_2)} \in T$. Then

$$\frac{(m_1, m_2)}{(1, 1)} = \frac{(m_1, m_2)}{(s_1, s_2)} \cdot \frac{(s_1, s_2)}{(1, 1)} \in T,$$

so $(m_1, m_2) \in T^c$, and hence $\lambda \in (S_1 \times S_2)^{-1}(T^c)$.

(3) Let $(m_1, m_2) \in (T^c)(S_1 \times S_2)$. Then there exists $(s_1, s_2) \in S_1 \times S_2$ with

$$(s_1, s_2)(m_1, m_2) \in T^c,$$

so

$$\frac{(s_1m_1, s_2m_2)}{(1, 1)} \in T.$$

Since $\frac{(1,1)}{(s_1,s_2)} \in (S_1 \times S_2)^{-1}(R_1 \times R_2)$ and T is a submodule of $(S_1 \times S_2)^{-1}(M_1 \times M_2)$,

$$\frac{(m_1, m_2)}{(1, 1)} = \frac{(1, 1)}{(s_1, s_2)} \cdot \frac{(s_1m_1, s_2m_2)}{(1, 1)} \in T.$$

Hence $(m_1, m_2) \in T^c$, which proves $(T^c)(S_1 \times S_2) \subseteq T^c$. The reverse inclusion is trivial. □

Now let $\mathcal{S}(M_1 \times M_2)$ and $\mathcal{S}(M_i)$, $i = 1, 2$, denote the sets of all submodules of $M_1 \times M_2$ and M_i, respectively. Let

$$\phi : \mathcal{S}(M_1 \times M_2) \longrightarrow \mathcal{S}(M_1 \times M_2) \cup \{\varnothing\}$$

and

$$\phi_i : \mathcal{S}(M_i) \longrightarrow \mathcal{S}(M_i) \cup \{\varnothing\}, \qquad i = 1, 2,$$

be functions with $\phi = \phi_1 \times \phi_2$, and assume $\phi(N) \subseteq N$ for every nonempty $\phi(N)$. We define

$$\phi_{S_1\times S_2} : \mathcal{S}\big((S_1 \times S_2)^{-1}(M_1 \times M_2)\big) \longrightarrow \mathcal{S}\big((S_1 \times S_2)^{-1}(M_1 \times M_2)\big) \cup \{\varnothing\}$$

by

$$\phi_{S_1\times S_2}(T) = (S_1 \times S_2)^{-1}\big(\phi(T^c)\big)$$

for every submodule $T \leq (S_1 \times S_2)^{-1}(M_1 \times M_2)$, where T^c is the contraction from Definition 4.5. By Lemma 4.6, T^c is uniquely determined by T, and $T = (S_1 \times S_2)^{-1}(T^c)$. Hence $\phi_{S_1\times S_2}$ is well-defined. Since $\phi(T^c) \subseteq T^c$, we have

$$\phi_{S_1\times S_2}(T) = (S_1 \times S_2)^{-1}\big(\phi(T^c)\big) \subseteq (S_1 \times S_2)^{-1}(T^c) = T.$$

Definition 4.7. A proper submodule $T = (S_1 \times S_2)^{-1}(N_1 \times N_2)$ of $(S_1 \times S_2)^{-1}(M_1 \times M_2)$ is called a $\phi_{S_1 \times S_2}$-*prime submodule* if whenever

$$\frac{(r_1, r_2)}{(s_1, s_2)} \in (S_1 \times S_2)^{-1}(R_1 \times R_2), \qquad \frac{(m_1, m_2)}{(t_1, t_2)} \in (S_1 \times S_2)^{-1}(M_1 \times M_2)$$

satisfy

$$\frac{(r_1, r_2)}{(s_1, s_2)} \frac{(m_1, m_2)}{(t_1, t_2)} \in T \setminus \phi_{S_1 \times S_2}(T),$$

it follows that

$$\frac{(r_1, r_2)}{(s_1, s_2)} \in (T :_{(S_1 \times S_2)^{-1}(R_1 \times R_2)} (S_1 \times S_2)^{-1}(M_1 \times M_2))$$

or $\frac{(m_1, m_2)}{(t_1, t_2)} \in T$.

The following lemma is the submodule analogue of Lemma 4.1.

Lemma 4.8 (Classification of ϕ-prime submodules). Let

$$\phi_i : \mathcal{S}(M_i) \longrightarrow \mathcal{S}(M_i) \cup \{\varnothing\}, \qquad i = 1, 2,$$

and let

$$\phi : \mathcal{S}(M_1 \times M_2) \longrightarrow \mathcal{S}(M_1 \times M_2) \cup \{\varnothing\}$$

be the function defined by $\phi = \phi_1 \times \phi_2$, with $\phi_i(N_i) \subseteq N_i$. Then the ϕ-prime submodules of $M_1 \times M_2$ are of exactly one of the following three types:

(1) $N_1 \times N_2$, where N_i is a proper submodule of M_i with $\phi_i(N_i) = N_i$ for $i = 1, 2$;
(2) $N_1 \times M_2$, where N_1 is a ϕ_1-prime submodule of M_1 which must be prime if $\phi_2(M_2) \neq M_2$;
(3) $M_1 \times N_2$, where N_2 is a ϕ_2-prime submodule of M_2 which must be prime if $\phi_1(M_1) \neq M_1$.

Proof . We adapt the argument of [11, Theorem 2.13] to the product setting. Let $N = N_1 \times N_2$ be a proper ϕ-prime submodule of $M_1 \times M_2$, where $\phi = \phi_1 \times \phi_2$. We distinguish three cases according to the behaviour of N under the projections ρ_1, ρ_2 of Remark 3.15.

Type (1): $N_1 \neq M_1$ *and* $N_2 \neq M_2$. We claim that in this case $\phi_i(N_i) = N_i$ for $i = 1, 2$. Suppose, for instance, that $\phi_1(N_1) \neq N_1$. Choose $x_1 \in N_1 \setminus \phi_1(N_1)$ and $x_2 \in M_2 \setminus N_2$. Then

$$(1_{R_1}, 0)(x_1, x_2) = (x_1, 0) \in N_1 \times N_2 = N.$$

Since $x_1 \notin \phi_1(N_1)$, we have

$$(x_1, 0) \notin \phi_1(N_1) \times \phi_2(N_2) = \phi(N).$$

Thus

$$(1_{R_1}, 0)(x_1, x_2) \in N \setminus \phi(N).$$

But $(1_{R_1}, 0) \notin (N :_{R_1 \times R_2} M_1 \times M_2)$, because $N_1 \neq M_1$, and $(x_1, x_2) \notin N$, because $x_2 \notin N_2$. This contradicts the ϕ-primality of N. Hence $\phi_1(N_1) = N_1$, and symmetrically $\phi_2(N_2) = N_2$. This gives Type (1).

Type (2): $N_2 = M_2$ *and* $N_1 \neq M_1$. We first show that N_1 is a ϕ_1-prime submodule of M_1. Let $r_1 \in R_1$ and $m_1 \in M_1$ satisfy $r_1 m_1 \in N_1 \setminus \phi_1(N_1)$. Then

$$(r_1, 1_{R_2})(m_1, m_2) = (r_1 m_1, m_2) \in N_1 \times M_2 = N$$

for every $m_2 \in M_2$. Moreover, $(r_1 m_1, m_2) \notin \phi_1(N_1) \times \phi_2(M_2) = \phi(N)$, because $r_1 m_1 \notin \phi_1(N_1)$. By the ϕ-primality of N, either $(r_1, 1_{R_2}) \in (N :_{R_1 \times R_2} M_1 \times M_2)$ or $(m_1, m_2) \in N$. The first alternative gives $r_1 \in (N_1 :_{R_1} M_1)$, while the second gives $m_1 \in N_1$. Hence N_1 is ϕ_1-prime.

Now suppose in addition that $\phi_2(M_2) \neq M_2$. We must show that N_1 is actually prime. Let $r_1, m_1 \in R_1, M_1$ with $r_1 m_1 \in N_1$. If $r_1 m_1 \notin \phi_1(N_1)$, then $r_1 \in (N_1 : M_1)$ or $m_1 \in N_1$ by ϕ_1-primality. If $r_1 m_1 \in \phi_1(N_1)$, choose $y_2 \in M_2 \setminus \phi_2(M_2)$. Then

$$(r_1, 1_{R_2})(m_1, y_2) = (r_1 m_1, y_2) \in N_1 \times M_2 = N.$$

Since $r_1 m_1 \in \phi_1(N_1)$ but $y_2 \notin \phi_2(M_2)$, we have

$$(r_1 m_1, y_2) \notin \phi_1(N_1) \times \phi_2(M_2) = \phi(N).$$

By the ϕ-primality of N, either $(r_1, 1_{R_2}) \in (N :_{R_1 \times R_2} M_1 \times M_2)$ or $(m_1, y_2) \in N$. The first gives $r_1 \in (N_1 :_{R_1} M_1)$, and the second gives $m_1 \in N_1$. Thus N_1 is prime. This gives Type (2).

Type (3): $N_1 = M_1$ *and* $N_2 \neq M_2$. The argument is symmetric to the previous case and yields that N_2 is ϕ_2-prime, and prime if $\phi_1(M_1) \neq M_1$.

Conversely, it is straightforward to verify that submodules of the three listed forms are ϕ-prime: for Type (1), the condition $\phi_i(N_i) = N_i$ makes the set $N \setminus \phi(N)$ empty, so the defining implication holds vacuously; for Types (2) and (3), the verification reduces to the ϕ_i-primality of N_i together with the stated primeness condition. This completes the proof. □

The one-factor localization criterion for ϕ-prime submodules is already available in Zamani [11, Theorem 2.12(ii)]. The next theorem records the corresponding coordinate-wise statement for product modules. In contrast to an earlier formulation, the properness of the localized factor is stated explicitly; this hypothesis is essential, as Example 4.10 below shows.

Theorem 4.9 (Transfer of ϕ-prime submodules). Let S_i be a multiplicatively closed subset of R_i, let N_1 be a ϕ_1-prime submodule of M_1, and put $K = N_1(S_1)$. Assume that

$$S_1^{-1}N_1 \subsetneq S_1^{-1}M_1, \tag{4.1}$$

$$S_1^{-1}\phi_1(N_1) \subseteq S_1^{-1}\phi_1(K), \tag{4.2}$$

and

$$S_2^{-1}\phi_2\big(M_2(S_2)\big) = S_2^{-1}M_2. \tag{4.3}$$

Then $T = S_1^{-1}N_1 \times S_2^{-1}M_2$ is a $\phi_{S_1\times S_2}$-prime submodule of $S_1^{-1}M_1 \times S_2^{-1}M_2$.

Symmetrically, if N_2 is a ϕ_2-prime submodule of M_2, $K_2 = N_2(S_2)$,

$$S_2^{-1}N_2 \subsetneq S_2^{-1}M_2, \quad S_2^{-1}\phi_2(N_2) \subseteq S_2^{-1}\phi_2(K_2), \quad S_1^{-1}\phi_1\big(M_1(S_1)\big) = S_1^{-1}M_1,$$

then $S_1^{-1}M_1 \times S_2^{-1}N_2$ is $\phi_{S_1\times S_2}$-prime.

Proof . We prove the first assertion. By (4.1), T is a proper submodule of $S_1^{-1}M_1 \times S_2^{-1}M_2$. Since $K = N_1(S_1)$, the contraction of $T = S_1^{-1}N_1 \times S_2^{-1}M_2$ is $K \times M_2(S_2)$. Hence, by the definition of the induced operation,

$$\phi_{S_1\times S_2}(T) = S_1^{-1}\phi_1(K) \times S_2^{-1}\phi_2\big(M_2(S_2)\big) = S_1^{-1}\phi_1(K) \times S_2^{-1}M_2,$$

where the last equality follows from (4.3). Thus

$$T \setminus \phi_{S_1\times S_2}(T) = (S_1^{-1}N_1 \setminus S_1^{-1}\phi_1(K)) \times S_2^{-1}M_2.$$

Take $a_i \in S_i^{-1}R_i$ and $x_i \in S_i^{-1}M_i$ such that $(a_1, a_2)(x_1, x_2) \in T \setminus \phi_{S_1\times S_2}(T)$. Write $a_1 = r_1/s_1$ and $x_1 = m_1/t_1$, with $r_1 \in R_1$, $m_1 \in M_1$, and $s_1, t_1 \in S_1$. Then

$$\frac{r_1m_1}{s_1t_1} \in S_1^{-1}N_1 \quad \text{and} \quad \frac{r_1m_1}{s_1t_1} \notin S_1^{-1}\phi_1(K).$$

The first membership gives some $u \in S_1$ such that $ur_1m_1 \in N_1$. Moreover, by (4.2),

$$\frac{r_1m_1}{s_1t_1} \notin S_1^{-1}\phi_1(N_1),$$

so in particular $ur_1m_1 \notin \phi_1(N_1)$. Therefore $ur_1m_1 \in N_1 \setminus \phi_1(N_1)$. Since N_1 is ϕ_1-prime, either $ur_1 \in (N_1 :_{R_1} M_1)$ or $m_1 \in N_1$. In the first case, $r_1/s_1 = (ur_1)/(us_1)$ belongs to $S_1^{-1}(N_1 :_{R_1} M_1)$, and hence a_1 acts on $S_1^{-1}M_1$ with image in $S_1^{-1}N_1$. Consequently

$$(a_1, a_2) \in (T :_{S_1^{-1}R_1\times S_2^{-1}R_2} S_1^{-1}M_1 \times S_2^{-1}M_2).$$

In the second case, $x_1 = m_1/t_1 \in S_1^{-1}N_1$, so $(x_1, x_2) \in T$. This proves that T is $\phi_{S_1\times S_2}$-prime.

The second assertion follows by interchanging the indices 1 and 2. □

Example 4.10 (Properness cannot be omitted). Let

$$R_1 = \mathbb{Z}, \qquad M_1 = \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}, \qquad S_1 = \mathbb{Z} \setminus \{0\}, \qquad N_1 = 0, \qquad \phi_1 \equiv 0,$$

and let

$$R_2 = M_2 = \mathbb{Z}, \qquad S_2 = \{1\}, \qquad \phi_2(M_2) = M_2.$$

Then N_1 is ϕ_1-prime, since $N_1 \setminus \phi_1(N_1) = \varnothing$, and $(N_1 :_{R_1} M_1) = \mathrm{Ann}_{\mathbb{Z}}(M_1) = 0$, so

$$S_1 \cap (N_1 :_{R_1} M_1) = \varnothing.$$

The compatibility condition (4.2) holds because both sides are zero, and the fullness condition (4.3) holds because $S_2^{-1}\phi_2(M_2(S_2)) = S_2^{-1}M_2 = M_2$.

However, $S_1^{-1}M_1 = M_1 \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, hence $S_1^{-1}N_1 = S_1^{-1}M_1 = 0$. Thus the properness condition (4.1) fails. In this case, $T = S_1^{-1}N_1 \times S_2^{-1}M_2 = 0 \times \mathbb{Z}$ equals the whole localized module $S_1^{-1}M_1 \times S_2^{-1}M_2 = 0 \times \mathbb{Z}$, so T is not proper and cannot be $\phi_{S_1\times S_2}$-prime. This shows that the properness condition (4.1) is essential and cannot be removed.

Example 4.11 (The fullness condition cannot be omitted). Let $R_1 = M_1 = \mathbb{Z}/6\mathbb{Z}$ and $R_2 = M_2 = \mathbb{Z}$, and take $S_1 = S_2 = \{1\}$. Define $\phi_i(L) = 0$ for every submodule L of M_i, and put $N_1 = 0$. Then N_1 is ϕ_1-prime, since $N_1 \setminus \phi_1(N_1) = \varnothing$, and $S_1 \cap (N_1 :_{R_1} M_1) = \varnothing$. The compatibility condition (4.2) also holds, and $S_1^{-1}N_1 = 0 \subsetneq M_1 = S_1^{-1}M_1$. However,

$$S_2^{-1}\phi_2(M_2(S_2)) = 0 \neq \mathbb{Z} = S_2^{-1}M_2,$$

so the fullness condition fails. In this case $T = 0 \times \mathbb{Z}$ and $\phi(T) = 0 \times 0$. The elements $a = (\overline{2}, 1)$ and $x = (\overline{3}, 1)$ satisfy $ax = (\overline{0}, 1) \in T \setminus \phi(T)$, whereas $a \notin (T :_{R_1 \times R_2} M_1 \times M_2)$ and $x \notin T$. Thus T is not ϕ-prime. This shows that a condition forcing the induced ϕ-operation to be full on the complementary factor is necessary for the stated transfer conclusion.

The fullness assumption (4.3) is essential to this argument: without it, non-membership in the induced ϕ-part may be caused solely by the second coordinate and gives no information about the first coordinate.

The following theorem provides a converse to Theorem 4.9 in terms of the *saturation* $N_1(S_1)$, rather than N_1 itself. As Example 4.13 below shows, the direct converse (i.e., without saturation) fails in general.

Theorem 4.12 (Converse via saturation). Let $T = (S_1 \times S_2)^{-1}(N_1 \times M_2)$ be a $\phi_{S_1 \times S_2}$-prime submodule of $(S_1 \times S_2)^{-1}(M_1 \times M_2)$, and put $K = N_1(S_1)$. Assume that $\phi_1(K)$ is S_1-saturated, in the sense that

$$\{m \in M_1 : \exists s \in S_1,\ sm \in \phi_1(K)\} = \phi_1(K),$$

where the equality is also understood when $\phi_1(K) = \varnothing$. Then $K = N_1(S_1)$ is a ϕ_1-prime submodule of M_1. Indeed, since $T^c = K \times M_2(S_2)$, the definition of the induced operation automatically gives

$$\phi_{S_1 \times S_2}(T) = (S_1 \times S_2)^{-1}(\phi_1(K) \times \phi_2(M_2(S_2))).$$

No additional compatibility assumption is needed.

Proof . The contraction of T is $T^c = K \times M_2(S_2)$. Suppose, to the contrary, that K is not ϕ_1-prime. Then there are $r_1 \in R_1$ and $m_1 \in M_1$ such that

$$r_1 m_1 \in K \setminus \phi_1(K), \qquad r_1 \notin (K :_{R_1} M_1), \qquad m_1 \notin K.$$

Set $x = \frac{(r_1,1_{R_2})}{(1,1)}$ and $y = \frac{(m_1,0)}{(1,1)}$. Then $xy = \frac{(r_1m_1,0)}{(1,1)} \in T$, because $r_1m_1 \in K$. Moreover, the S_1-saturation assumption on $\phi_1(K)$ implies $\frac{r_1m_1}{1} \notin S_1^{-1}\phi_1(K)$. Consequently,

$$xy \notin (S_1 \times S_2)^{-1}\big(\phi_1(K) \times \phi_2(M_2(S_2))\big) = \phi_{S_1\times S_2}(T),$$

so $xy \in T \setminus \phi_{S_1\times S_2}(T)$. By the $\phi_{S_1\times S_2}$-primality of T, either

$$x \in (T :_{(S_1\times S_2)^{-1}(R_1\times R_2)} (S_1 \times S_2)^{-1}(M_1 \times M_2)) \quad \text{or} \quad y \in T.$$

The latter implies $(m_1, 0) \in T^c = K \times M_2(S_2)$, hence $m_1 \in K$, a contradiction.

If the former holds, then for every $m_1' \in M_1$,

$$x\frac{(m_1', 0)}{(1, 1)} = \frac{(r_1m_1', 0)}{(1, 1)} \in T.$$

Thus $(r_1m_1', 0) \in T^c = K \times M_2(S_2)$, so $r_1m_1' \in K$ for every $m_1' \in M_1$. This means $r_1 \in (K :_{R_1} M_1)$, again a contradiction. Therefore K is ϕ_1-prime. □

Example 4.13 (The direct converse fails). Let $R_1 = R_2 = \mathbb{Z}$ and $S_1 = S_2 = \mathbb{Z} \setminus \{0\}$. Let $M_1 = \mathbb{Z}$ and $M_2 = \mathbb{Z}/6\mathbb{Z}$. Consider the submodule $N = N_1 \times N_2 = 0 \times 0$ of $M_1 \times M_2$. Then N is not a prime (and hence not a ϕ-prime for $\phi = 0$) submodule of $M_1 \times M_2$, because

$$(2, \overline{3})(0, \overline{2}) = (0, \overline{0}) \in N,$$

but

$$(2, \overline{3}) \notin (N :_{R_1\times R_2} M_1 \times M_2) \quad \text{and} \quad (0, \overline{2}) \notin N.$$

However, the localization

$$(S_1 \times S_2)^{-1}(M_1 \times M_2) \cong \mathbb{Q} \times 0 \cong \mathbb{Q}$$

is a field, and $(S_1 \times S_2)^{-1}(N) = 0$ is a prime (hence ϕ-prime) submodule of $\mathbb{Q}$.

On the other hand, the saturation of N with respect to $S_1 \times S_2$ is

$$(N_1 \times N_2)(S_1 \times S_2) = N_1(S_1) \times N_2(S_2) = 0 \times \mathbb{Z}/6\mathbb{Z}.$$

Then $M_1 \times M_2/(0 \times \mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}$, which is a domain, implying that $0 \times \mathbb{Z}/6\mathbb{Z}$ is indeed a prime (hence ϕ-prime) submodule of $M_1 \times M_2$. Thus, it is the *saturation* of N, rather than

N itself, that is preserved as a prime submodule under localization. This is consistent with Theorem 4.12.

The mechanism behind this example is that the S_2-torsion part of M_2 is killed by the localization, whereas the S_1-torsion part of M_1 is zero; consequently, the localized product module collapses to $\mathbb{Q}$, in which the localized submodule 0 is automatically prime. The saturation $N_1(S_1) \times N_2(S_2)$ records precisely the information that survives this collapse.

Example 4.14. Let $P = (p)$ and $Q = (q)$ be two prime ideals of $\mathbb{Z}$, where p and q are distinct prime numbers. Consider the multiplicatively closed sets

$$S_P = \mathbb{Z} \setminus (p), \qquad S_Q = \mathbb{Z} \setminus (q).$$

The corresponding localizations are denoted by

$$\mathbb{Z}_P = \mathbb{Z}_{(p)}, \qquad \mathbb{Z}_Q = \mathbb{Z}_{(q)}.$$

Thus,

$$\mathbb{Z}_{(p)} = \left\{ \frac{m}{n} : \ m, n \in \mathbb{Z}, \ p \nmid n \right\}, \qquad \mathbb{Z}_{(q)} = \left\{ \frac{l}{t} : \ l, t \in \mathbb{Z}, \ q \nmid t \right\}.$$

Since $(p)\mathbb{Z}_{(p)} \in \operatorname{Spec}(\mathbb{Z}_{(p)})$ and $(q)\mathbb{Z}_{(q)} \in \operatorname{Spec}(\mathbb{Z}_{(q)})$, it follows that

$$(p)\mathbb{Z}_{(p)} \times \mathbb{Z}_{(q)} \quad \text{and} \quad \mathbb{Z}_{(p)} \times (q)\mathbb{Z}_{(q)}$$

are prime ideals of $\mathbb{Z}_{(p)} \times \mathbb{Z}_{(q)}$.

Example 4.15. Let

$$S_1 = \{1, 2, 2^2, \dots\}, \qquad S_2 = \{1, 3, 3^2, \dots\}$$

be multiplicatively closed subsets of $\mathbb{Z}$. Consider the localized modules

$$M_1 = S_1^{-1}\mathbb{Z} = \left\{ \frac{m}{2^n} : \ m \in \mathbb{Z}, \ n \geq 0 \right\}, \qquad M_2 = S_2^{-1}\mathbb{Z} = \left\{ \frac{l}{3^t} : \ l \in \mathbb{Z}, \ t \geq 0 \right\}.$$

The submodules

$$P_1 = \left\{ \frac{3m}{2^n} : \ m \in \mathbb{Z}, \ n \geq 0 \right\}, \qquad P_2 = \left\{ \frac{5l}{3^t} : \ l \in \mathbb{Z}, \ t \geq 0 \right\}$$

are prime submodules of M_1 and M_2, respectively. Therefore,

$$P_1 \times S_2^{-1}\mathbb{Z} \quad \text{and} \quad S_1^{-1}\mathbb{Z} \times P_2$$

are prime submodules of the localized product module $S_1^{-1}\mathbb{Z} \times S_2^{-1}\mathbb{Z}$.

5 Conclusion and future work

We studied localization of product rings and modules at $S_1 \times S_2$. The main structural results describe the localization map, saturation and extension ideals, and identify the localized product ring with $S_1^{-1}R_1 \times S_2^{-1}R_2$. For modules, we established exactness and compatibility with quotients and colons, and proved preservation of multiplication modules. We also introduced the induced ϕ-operation through contraction and saturation and obtained transfer and converse results for ϕ-prime ideals and submodules. In the transfer theorem, the properness condition on the localized factor was stated explicitly; examples show that neither this condition nor the fullness condition on the complementary factor can be omitted.

Several questions remain. It would be useful to determine whether the saturation in Theorem 4.12 has an optimal characterization, to identify weaker hypotheses for the colon formula beyond finite generation, and to extend the ϕ-prime results to other generalized primality notions and to finite products. The noncommutative setting also merits further study.

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Conflict of interest
The author declares that they have no conflict of interest.

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