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CS-prime elements in a frame

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Abstract

In this paper we study completely strongly prime elements (*cs*-prime elements) of a frame, defined by means of arbitrary joins as a strengthening of the classical notion of a prime element. We first show that, for a frame, this notion coincides with the previously considered notion of a completely prime (*c*-prime) element, so that no genuinely new class of elements is obtained; we also exhibit, by means of an explicit example, a (non-distributive) lattice in which this coincidence fails, showing that the frame hypothesis cannot be dropped. The *cs*-prime formulation is, however, a convenient working characterization, which we use throughout the paper together with the classical lattice-theoretic notions of completely join-prime and completely join-irreducible elements. Using this characterization we obtain several equivalent descriptions of (completely) prime elements and we investigate their behaviour under various frame-theoretic constructions, namely frame homomorphisms, quotient frames, $\wedge$-factorization frames and product frames.

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1 Introduction

As algebraic structures, frames are a natural generalization of lattice of ideals of a ring, and it is natural to ask which properties of rings admit a counterpart for frames. The absence of subtraction shows that many ring-theoretic results have no direct frame analogue and must be reformulated; nonetheless, there are many structural similarities between the elements of a frame and the ideals of a ring. Prime elements and their generalizations play an important role in understanding the structure of a frame, just as prime ideals do for a ring. *By a frame $\mathcal{L}$ we mean a complete lattice in which the meet distributes over arbitrary joins.*

The notion of prime elements has a significant place in the theory of rings, and it is used to characterize certain classes of rings. For years, there have been many studies and generalizations on this topic. Therefore, a thorough understanding of prime elements is indispensable for working in this theory. Let M be a module over a commutative ring R. An R-submodule N of M is said to be irreducible if N is not the intersection of two submodules of M that properly contain it. An ideal I of R is irreducible if it is irreducible as a submodule of the R-module R. Heinzer et al. [8] generalized the concept of irreducible ideals as follows: a proper ideal I of R is strongly irreducible if, for ideals J and K of R, $J \cap K \subseteq I$ implies $J \subseteq I$ or $K \subseteq I$. The notion of strongly irreducible submodules was introduced and studied in [3]. A submodule N of an R-module M is strongly irreducible if, for submodules N_1 and N_2 of M, the inclusion $N_1 \cap N_2 \subseteq N$ implies that either $N_1 \subseteq N$ or $N_2 \subseteq N$. A non-zero submodule N of M is strongly hollow in M if, for any submodules N_1 and N_2 of M, $N \subseteq N_1 + N_2$ implies $N \subseteq N_1$ or $N \subseteq N_2$. Also, a non-zero submodule N of M is completely hollow in M if, for any non-empty family $\{N_i\}_{i\in J}$ of submodules of M, $N = \sum_{i\in J} N_i$ implies that $N = N_j$ for some $j \in J$. A non-zero ideal I of R is strongly hollow (resp. completely hollow) whenever I is a strongly hollow (resp. completely hollow) submodule of the R-module R. The notion of strongly hollow submodules was introduced and studied by Abuhlail and Lomp in [1] as a dual notion of strongly irreducible submodules. The notion of strongly hollow ideals (resp. strongly hollow elements) was introduced and studied by Rostami in [12] (see also [4]).

With this motivation, we consider the concept of a completely strongly prime element (briefly, cs-prime element) in a frame as a generalization of a prime element. A proper element p of a frame $\mathcal{L}$ is called prime if $p \leq x \vee y$ implies $p \leq x$ or $p \leq y$ [5]. In this dictionary, a prime element of $\mathcal{L}$ is precisely the frame-theoretic counterpart of a strongly hollow submodule, a c-prime element (Definition 3.1(1)) corresponds to a completely hollow submodule, and, dually, the notion of a prime ideal of $\mathcal{L}$ used throughout this paper (Preliminaries) is the counterpart of a strongly irreducible ideal or submodule.

Among many other results in this paper, the first, preliminaries section contains elementary observations needed later on. In the third section, we collect some basic properties

of cs-prime elements. First, we give the definition of cs-prime elements (Definition 3.1) and provide conditions under which the join of two cs-prime elements is a cs-prime element (Proposition 3.6). In Proposition 3.7, it is shown that, in general, every atom element is cs-prime. We also exhibit (Remark 3.2 and Examples 3.3, 3.4) a non-distributive lattice in which a c-prime element need not be cs-prime, showing that the frame hypothesis in Proposition 3.5(1) is essential. Some basic equivalent characterizations of cs-prime elements for any frame are presented in Theorems 3.9, 3.10, 3.11, 3.14, and 3.17. It is proved in Theorem 3.16 that if p is a cs-prime element, then there is at most one coatom element c such that $p \not\leq c$. In Theorem 3.19, it is shown that if p is cs-prime, then $\pi_p = \{x \in \mathcal{L} : p \not\leq x\}$ is a prime ideal.

We continue in Section 4 by investigating the stability of cs-prime elements in various frame-theoretic constructions. In particular, we investigate the behavior of cs-prime elements under frame homomorphisms (Theorem 4.1), in factor frames (Propositions 4.3 and 4.5), in $\wedge$-factorization frames (Proposition 4.9), and in cartesian product frames (Propositions 4.10 and 4.12). Throughout Section 4, we freely use several classical facts of general lattice theory concerning the construction of a quotient lattice $\mathcal{L}/I$ by an ideal I and the correspondence between ideals of $\mathcal{L}$ containing I and ideals of $\mathcal{L}/I$, for which we refer to Grätzer and Schmidt [6], Grätzer [7], and Clugreanu [2]; the standard facts on frame homomorphisms and product frames used there may be found in Johnstone [10] and Picado and Pultr [11].

2 Preliminaries

Let $\mathcal{L}$ be a frame, and let $a, b \in \mathcal{L}$ with $a \leq b$. The sublattice

$$\{x \in \mathcal{L} : a \leq x \leq b\}$$

is called a quotient sublattice and is denoted by b/a. An element x of a lattice $\mathcal{L}$ is called an atom (resp. coatom) if there is no $y \in \mathcal{L}$ such that $0 < y < x$ (resp. $x < y < 1$). The sets of all coatoms and atoms of $\mathcal{L}$ are denoted by $\mathcal{CA}(\mathcal{L})$ and $\mathcal{A}(\mathcal{L})$, respectively. The meet of all coatoms of $\mathcal{L}$, denoted by $J(\mathcal{L})$, is called the Jacobson radical of the lattice $\mathcal{L}$; that is,

$$J(\mathcal{L}) = \bigwedge_{c \in \mathcal{CA}(\mathcal{L})} c.$$

An element x of a lattice $\mathcal{L}$ is called nontrivial (resp. proper) if $x \neq 0, 1$ (resp. $x \neq 1$). A non-empty subset I of a frame $\mathcal{L}$ is called an ideal if, for every $a \in I$ and $b \in \mathcal{L}$, $b \leq a$ implies that $b \in I$, and $x \vee y \in I$ for all $x, y \in I$. A proper ideal P of $\mathcal{L}$ is called prime if

$$x \wedge y \in P \quad \Longrightarrow \quad x \in P \text{ or } y \in P$$

for all $x, y \in \mathcal{L}$.

If $\mathcal{L}$ and $\mathcal{L}'$ are frames, a frame homomorphism $v : \mathcal{L} \to \mathcal{L}'$ is a map that preserves arbitrary joins and finite meets (see [10, 11]); that is,

$$v\left(\bigvee S\right) = \bigvee_{x \in S} v(x), \qquad v(x \wedge y) = v(x) \wedge v(y), \qquad v(1) = 1,$$

for all $x, y \in \mathcal{L}$ and $S \subseteq \mathcal{L}$. Note that $v(0) = 0$ follows by taking $S = \emptyset$.

A map $v : P \to P'$ between posets P and P' is called an order embedding if

$$x \leq y \quad \text{if and only if} \quad v(x) \leq v(y)$$

for all $x, y \in P$. We need the following elementary observation.

Lemma 2.1. A non-empty subset I of a frame $\mathcal{L}$ is an ideal if and only if

$$x \wedge z \in I \quad \text{and} \quad x \vee y \in I$$

for all $x, y \in I$ and $z \in \mathcal{L}$. Moreover, since

$$x = x \wedge (x \vee y) \quad \text{and} \quad y = y \wedge (x \vee y),$$

if $x \vee y \in I$, then $x, y \in I$.

Any unexplained terminology related to lattice theory can be found in [2].

3 *CS*-prime Elements

Let $\mathcal{L}$ be a frame. In this section, we collect some basic properties of cs-prime elements of $\mathcal{L}$. We begin with the following definition.

Definition 3.1. Let p be a nonzero element of a frame $\mathcal{L}$.

(1) The element p is called a completely prime element, or briefly, a c-prime element, in $\mathcal{L}$ if, for every non-empty subset S of $\mathcal{L}$, the equality

$$p = \bigvee S$$

implies that $p = s$ for some $s \in S$.

(2) The element p is called a completely strongly prime element, or briefly, a cs-prime element, in $\mathcal{L}$ if, for every non-empty subset S of $\mathcal{L}$,

$$p \leq \bigvee S$$

implies that $p \leq s$ for some $s \in S$.

In the terminology of general lattice theory, condition (1) says precisely that p is completely join-irreducible, whereas condition (2) says that p is completely join-prime. In a general complete lattice, every completely join-prime element is completely join-irreducible, but the converse need not hold.

Remark 3.2. The hypothesis that $\mathcal{L}$ is a *frame*, that is, that finite meets distribute over arbitrary joins, is essential in Proposition 3.5(1). In a general lattice, every cs-prime element is c-prime, but a c-prime element need not be cs-prime. Examples 3.3 and 3.4 below exhibit a non-distributive, and hence non-frame, lattice in which this implication fails.

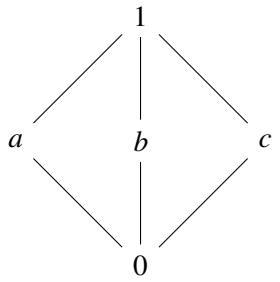


Figure 1: The non-distributive lattice $M_3 = \{0, a, b, c, 1\}$, with $a \wedge b = a \wedge c = b \wedge c = 0$ and $a \vee b = a \vee c = b \vee c = 1$.

Example 3.3. Let $\mathcal{L} = M_3 = \{0, a, b, c, 1\}$ be the lattice with $a \wedge b = a \wedge c = b \wedge c = 0$ and $a \vee b = a \vee c = b \vee c = 1$. Note that $\mathcal{L}$ is *not* a frame, since

$$a \wedge (b \vee c) = a \wedge 1 = a \neq 0 = (a \wedge b) \vee (a \wedge c).$$

The only elements of $\mathcal{L}$ below a are 0 and a itself. If $S \subseteq \mathcal{L}$ is a non-empty subset with $a = \bigvee S$, then $S \subseteq \{0, a\}$; since $\bigvee\{0\} = 0 \neq a$, we must have $a \in S$, so $a = s$ for $s = a \in S$. Hence a is c-prime.

Take $S = \{b, c\}$. Then $\bigvee S = b \vee c = 1 \geq a$, so $a \leq \bigvee S$. However, $a \not\leq b$ and $a \not\leq c$, so there is no $s \in S$ with $a \leq s$. Hence a is not cs-prime. Therefore, a is c-prime but not cs-prime in $\mathcal{L}$.

Example 3.4. (A concrete realization of Example 3.3.) Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, x, y, xy\}$ and let $\mathcal{L}$ be the lattice of subgroups of G under inclusion, with $H_1 \wedge H_2 = H_1 \cap H_2$ and $H_1 \vee H_2 = \langle H_1 \cup H_2 \rangle$. Then

$$\mathcal{L} = \{\{e\}, \langle x \rangle, \langle y \rangle, \langle xy \rangle, G\} \cong M_3,$$

via $0 \leftrightarrow \{e\}$, $a \leftrightarrow \langle x \rangle$, $b \leftrightarrow \langle y \rangle$, $c \leftrightarrow \langle xy \rangle$, $1 \leftrightarrow G$. As in Example 3.3, $\langle x \rangle$ is c-prime (being an atom, it can be written as a join of a set only if that set already contains $\langle x \rangle$) but not cs-prime, since

$$\langle x \rangle \leq \langle y \rangle \vee \langle xy \rangle = G,$$

while $\langle x \rangle \not\leq \langle y \rangle$ and $\langle x \rangle \not\leq \langle xy \rangle$.

The next proposition shows that, in a frame, the distributivity of finite meets over arbitrary joins forces the two notions to coincide. Thus, c-prime and cs-prime elements describe the same class of elements from two different, but equally natural, points of view.

Proposition 3.5. Let p be a nonzero element of a frame $\mathcal{L}$. Then:

(1) p is c-prime if and only if p is cs-prime.
(2) If p is c-prime (equivalently, cs-prime), then p is prime.

Proof . (1) Suppose first that p is cs-prime, and let

$$p = \bigvee S$$

for some non-empty subset S of $\mathcal{L}$. Since $p \leq \bigvee S$, there exists $t \in S$ such that $p \leq t$. On the other hand, $t \leq p$ because $p = \bigvee S$. Hence, $t = p$, and therefore p is c-prime.

Conversely, suppose that p is c-prime, and let

$$p \leq \bigvee S$$

for some non-empty subset S of $\mathcal{L}$. Since $\mathcal{L}$ is a frame,

$$p = p \wedge \bigvee S = \bigvee_{s \in S} (p \wedge s).$$

Since p is c-prime, there exists $s \in S$ such that

$$p = p \wedge s.$$

Thus, $p \leq s$, and hence p is cs-prime.

(2) Assume that p is cs-prime, and let $a, b \in \mathcal{L}$ satisfy

$$p \leq a \vee b.$$

Applying Definition 3.1(2) to $S = \{a, b\}$, we obtain

$$p \leq a \quad \text{or} \quad p \leq b.$$

Therefore, p is prime. □

Proposition 3.5(1) shows that a cs-prime element is nothing other than a c-prime element, described using an inequality rather than an equality; equivalently, by the discussion preceding Proposition 3.5, completely join-prime and completely join-irreducible elements of a frame coincide. No new class of frame elements is introduced by Definition 3.1(2). What Definition 3.1(2) does provide is a formulation that is considerably easier to manipulate in proofs, since the hypothesis $p \leq \bigvee S$ is often easier to verify, and easier to combine with further inequalities, than the equality $p = \bigvee S$; for this reason, we work almost exclusively with the cs-prime formulation for the remainder of the paper, keeping in mind that every statement about cs-prime elements is, via Proposition 3.5(1), simultaneously a statement about c-prime elements. In particular, the characterizations obtained in Theorem 3.9, Theorem 3.10, Theorem 3.11, and Theorem 3.19 below should be read as further descriptions of this single notion (useful chiefly because of the technical role played by the ideal π_p in Section 4), while the results on joins and meets of cs-prime elements (Propositions 3.6, 3.7, and 3.12), the coatom and Jacobson radical description (Theorem 3.16), the $\wedge$-factorizations of Section 4, and the behaviour of these elements under the frame-theoretic constructions studied there constitute the genuinely new content of the paper.

Proposition 3.6. If p and q are two cs-prime elements of a frame $\mathcal{L}$, then $p \vee q$ is cs-prime if and only if $p \leq q$ or $q \leq p$.

Proof . Suppose that $p \vee q$ is cs-prime, and let $S = \{p, q\}$. Since $p \vee q \leq \bigvee S$, we conclude that either $p \vee q \leq p$ or $p \vee q \leq q$; in the first case, $q \leq p$, and in the second, $p \leq q$.

Conversely, suppose that $p \leq q$ (the case $q \leq p$ is entirely symmetric). Then $p \vee q = q$, which is cs-prime by hypothesis, so $p \vee q$ is cs-prime. □

Proposition 3.7. If p is an atom of a frame $\mathcal{L}$, then p is cs-prime.

Proof . Let S be a non-empty subset of $\mathcal{L}$ such that

$$p \leq \bigvee S.$$

Suppose, to the contrary, that $p \not\leq s$ for every $s \in S$. Since p is an atom, this implies

$$p \wedge s = 0$$

for every $s \in S$. Therefore,

$$p = p \wedge \bigvee_{s \in S} s = \bigvee_{s \in S} (p \wedge s) = 0,$$

which contradicts the assumption that $p \neq 0$. Hence, $p \leq s$ for some $s \in S$, and consequently, p is cs-prime. □

Lemma 3.8. Let p be a cs-prime element of a frame $\mathcal{L}$. If

$$\pi_p := \{x \in \mathcal{L} : p \not\leq x\},$$

then $\bigvee \pi_p \in \pi_p$.

Proof . On the contrary, assume that

$$\bigvee \pi_p = \bigvee_{e \in \pi_p} e \notin \pi_p,$$

so $p \leq \bigvee \pi_p$. Since p is a cs-prime element, we conclude that there exists $e \in \pi_p$ such that $p \leq e$, a contradiction. Thus, $\bigvee \pi_p \in \pi_p$. □

Theorem 3.9. If p is a nonzero element of a frame $\mathcal{L}$, then p is a cs-prime element if and only if π_p is an ideal of $\mathcal{L}$.

Proof . Assume that p is a cs-prime element, and let $a, b \in \pi_p$ (so $p \not\leq a$ and $p \not\leq b$). Set $S = \{a, b\}$. If $p \leq a \vee b = \bigvee S$, then, since p is cs-prime, we have

$$p \leq a \quad \text{or} \quad p \leq b,$$

which is impossible. Therefore, $p \not\leq a \vee b$, and hence $a \vee b \in \pi_p$.

Next, let $a \in \pi_p$ (so $p \not\leq a$) and $c \in \mathcal{L}$. If $p \leq a \wedge c$, then $p \leq a$, a contradiction. Therefore, $a \wedge c \in \pi_p$; hence, π_p is an ideal of $\mathcal{L}$ by Lemma 2.1.

Conversely, assume that π_p is an ideal, and suppose to the contrary that $p \leq \bigvee S$ for some subset S of $\mathcal{L}$ with $p \not\leq s$ for all $s \in S$. This implies that $S \subseteq \pi_p$, and hence $\bigvee S \leq \bigvee \pi_p \in \pi_p$ by Lemma 3.8. Thus, $p \not\leq \bigvee S$, which is a contradiction. Therefore, p is a cs-prime element. □

Theorem 3.10. If $(\mathcal{L}, \leq)$ is a total ordering, then every nonzero element of the frame $\mathcal{L}$ is cs-prime.

Proof . Let p be a nonzero element of $\mathcal{L}$, and let $x, y \in \pi_p$. Since $\mathcal{L}$ is a total ordering, we have either $x \leq y$ or $y \leq x$, which implies $x \vee y \in \{x, y\} \subseteq \pi_p$. Moreover, for any $z \in \mathcal{L}$, since $p \not\leq x$ and $x \wedge z \leq x$, we must have $p \not\leq x \wedge z$, so $x \wedge z \in \pi_p$. Thus, π_p is an ideal of $\mathcal{L}$. This shows that p is a cs-prime element by Theorem 3.9. □

Theorem 3.11. If p is a nonzero element of a frame $\mathcal{L}$, then p is cs-prime if and only if $p \not\leq \bigvee \pi_p$.

Proof . Suppose that $p \not\leq \bigvee \pi_p$, and let $p \leq \bigvee S$ for some subset S of $\mathcal{L}$. If $p \not\leq s$ for all $s \in S$, then $S \subseteq \pi_p$, which implies $\bigvee S \leq \bigvee \pi_p$. Hence, $p \leq \bigvee \pi_p$, a contradiction. Therefore, there is an element $s \in S$ such that $p \leq s$, showing that p is cs-prime. The converse implication follows from Lemma 3.8. □

Proposition 3.12. Let $\mathcal{L}$ be a frame and $a, b \in \mathcal{L}$. Then $a \vee b$ is cs-prime in $\mathcal{L}$ if and only if either $a \leq b$ and b is cs-prime, or $b \leq a$ and a is cs-prime.

Proof . Suppose that $c = a \vee b$ is a cs-prime element in $\mathcal{L}$. Since $c \leq a \vee b = \bigvee\{a, b\}$, the cs-prime property of c implies that $c \leq a$ or $c \leq b$. If $c = a$, then $b \leq a$, and if $c = b$, then $a \leq b$.

Conversely, suppose that $a \leq b$ and b is cs-prime. Then $a \vee b = b$. If $a \vee b \leq \bigvee S$ for some subset S of $\mathcal{L}$, then $b \leq \bigvee S$. Since b is cs-prime, there exists $s \in S$ such that $b \leq s$, which means $a \vee b \leq s$. Thus, $a \vee b$ is cs-prime. Similarly, if $b \leq a$ and a is cs-prime, then $a \vee b = a$ is cs-prime, as required. □

The following example shows that the meet of two cs-prime elements need not be a cs-prime element.

Example 3.13. Let $\mathcal{L} = \{0, a, b, 1\}$ be a frame with the relations $0 \leq a \leq 1$, $0 \leq b \leq 1$, $a \vee b = 1$, and $a \wedge b = 0$. Clearly, the elements a and b are cs-prime elements, whereas their meet $a \wedge b = 0$ is not cs-prime. Therefore, the meet of two cs-prime elements need not be a cs-prime element.

We next give two other characterizations of cs-prime elements.

Theorem 3.14. Let p be a nonzero element of a frame $\mathcal{L}$. Then the following statements are equivalent:

(1) p is cs-prime;
(2) For any non-empty subset $S \subseteq \mathcal{L}$, if $p \leq \bigvee S$, then $p/0 \subseteq s/0$ for some $s \in S$;
(3) For any non-empty subset $S \subseteq \mathcal{L}$, if $p = \bigvee_{s\in S}(p \wedge s)$, then $p/0 \subseteq s/0$ for some $s \in S$.

Proof . (1) $\Rightarrow$ (2) Suppose that p is cs-prime and let S be a non-empty subset of $\mathcal{L}$ such that $p \leq \bigvee S$. Since p is cs-prime, there exists $s \in S$ such that $p \leq s$. Thus, for every $x \in p/0$, we have $x \leq p \leq s$, which implies $x \in s/0$. Hence, $p/0 \subseteq s/0$.

(2) $\Rightarrow$ (3) Let $S \subseteq \mathcal{L}$ such that $p = \bigvee_{s\in S}(p \wedge s)$. Then $p \leq \bigvee S$. By (2), there exists $s \in S$ such that $p/0 \subseteq s/0$.

(3) $\Rightarrow$ (1) Let S be a non-empty subset of $\mathcal{L}$ such that $p \leq \bigvee S$. By the distributivity of the frame $\mathcal{L}$, we have

$$p = p \wedge \bigvee S = \bigvee_{s\in S}(p \wedge s).$$

By (3), there exists $s \in S$ such that $p/0 \subseteq s/0$. Since $p \in p/0$, it follows that $p \in s/0$, which means $p \leq s$. Consequently, p is a cs-prime element of $\mathcal{L}$. □

Proposition 3.15. If p is a cs-prime element of a frame $\mathcal{L}$, then

$$I_p := \{x \in \mathcal{L} : x \lneqq p\}$$

is an ideal of $\mathcal{L}$ with a unique maximal element.

Proof . Clearly, $0 \in I_p$, so $I_p \neq \emptyset$. Let $x, y \in I_p$ and $z \in \mathcal{L}$. Then $x \lneqq p$ and $y \lneqq p$. Since p is cs-prime, we have $x \vee y \lneqq p$. Moreover, $x \wedge z \leq x \lneqq p$, which implies $x \vee y, x \wedge z \in I_p$. Therefore, I_p is an ideal of $\mathcal{L}$.

Next, we view $(I_p, \leq)$ as a poset and show that every chain in I_p has an upper bound in I_p. Let $C \subseteq I_p$ be a chain and define $m := \bigvee C$ (computed in $\mathcal{L}$). Since $c \leq p$ for all $c \in C$, p is an upper bound of C, whence $m \leq p$. If $m = p$, then $p \leq \bigvee C$. Since p is cs-prime, there exists $c \in C$ such that $p \leq c$. However, $c \in I_p$ entails $c \lneqq p$, yielding the contradiction $p \leq c < p$. Thus $m \neq p$, which together with $m \leq p$ gives $m \lneqq p$, i.e., $m \in I_p$. By Zorn's Lemma, I_p has a maximal element.

Now, suppose $b, b' \in I_p$ are two maximal elements with $b \neq b'$. By the maximality of b and b' in I_p, we must have $b \vee b' = p$; otherwise, $b \vee b' \in I_p$ with $b \lneqq b \vee b'$, which contradicts the maximality of b. Consequently, $p = b \vee b'$. Since p is cs-prime, we have $p \leq b$ or $p \leq b'$, that is, $p = b$ or $p = b'$, directly contradicting $b, b' \in I_p$. Hence, I_p contains exactly one maximal element. □

The next theorem shows that if p is a cs-prime element, then there is at most one coatom c such that $p \nleq c$.

Theorem 3.16. If p is a cs-prime element of a frame $\mathcal{L}$, then either $p \leq J(\mathcal{L})$ or there is exactly one coatom c such that $p \nleq c$.

Proof . Assume that $p \nleq J(\mathcal{L}) = \bigwedge_{c \in C\mathcal{A}} c$. Then there exists a coatom c of $\mathcal{L}$ such that $p \nleq c$. If c' is another coatom of $\mathcal{L}$ with $p \nleq c'$, then $c \vee c' = 1$. Consequently, $p \leq c \vee c'$ implies that either $p \leq c$ or $p \leq c'$, which is a contradiction. Hence, there is exactly one coatom c such that $p \nleq c$. □

Theorem 3.17. Let $\mathcal{B}$ be a subset of a frame $\mathcal{L}$ such that every element of $\mathcal{L}$ is a meet of elements of $\mathcal{B}$. Then p is cs-prime if and only if for all $a, b \in \mathcal{B}$, if $p \leq a \vee b$, then $p \leq a$ or $p \leq b$.

Proof . Suppose p is cs-prime, and let $a, b \in \mathcal{B}$ be such that $p \leq a \vee b$. Applying Definition 3.1(2) with $S = \{a, b\}$ gives $p \leq a$ or $p \leq b$.

Conversely, assume the stated condition on $\mathcal{B}$ holds. By Theorem 3.9, it suffices to show that

$$\pi_p := \{x \in \mathcal{L} : p \nleq x\}$$

is an ideal of $\mathcal{L}$. Since π_p is clearly downward closed, it remains to show that π_p is closed under finite joins. Let $x, y \in \pi_p$, so $p \nleq x$ and $p \nleq y$. By hypothesis, we can write $x = \bigwedge_{a \in C} a$ and $y = \bigwedge_{b \in D} b$ for some subsets $C, D \subseteq \mathcal{B}$. Since $p \nleq x = \bigwedge_{a \in C} a$, there exists $a \in C$ such that $p \nleq a$; likewise, there exists $b \in D$ such that $p \nleq b$. Now, $x \leq a$ and $y \leq b$. If $p \leq x \vee y$, then $p \leq a \vee b$. Applying the hypothesis to $a, b \in \mathcal{B}$ then yields $p \leq a$ or $p \leq b$, which contradicts the choice of a and b. Hence, $p \nleq x \vee y$, meaning that $x \vee y \in \pi_p$. Thus, π_p is an ideal of $\mathcal{L}$, and therefore p is cs-prime by Theorem 3.9. □

Lemma 3.18. If p is a cs-prime element of a frame $\mathcal{L}$, then

$$\Gamma_p := \{a \in \mathcal{L} : a \wedge p \in \pi_p\}$$

is an ideal of $\mathcal{L}$.

Proof . Let $a, b \in \Gamma_p$ and $c \in \mathcal{L}$. Then

$$(a \vee b) \wedge p = (p \wedge a) \vee (p \wedge b) \in \pi_p$$

and

$$(a \wedge c) \wedge p = (a \wedge p) \wedge c \in \pi_p$$

since π_p is an ideal of $\mathcal{L}$ (by Theorem 3.9). Thus, Γ_p is an ideal, as required. □

Theorem 3.19. If p is a cs-prime element of a frame $\mathcal{L}$, then the following hold:

(1) π_p is a prime ideal of $\mathcal{L}$;

(2) $\pi_p = \Gamma_p$.

Proof . (1) Let $a \wedge b \in \pi_p$, which means $p \nleq a \wedge b$. Suppose, for a contradiction, that $a \notin \pi_p$ and $b \notin \pi_p$. Then $p \leq a$ and $p \leq b$. By the property of the meet, this implies $p \leq a \wedge b$, which contradicts our assumption that $a \wedge b \in \pi_p$. Hence, $a \in \pi_p$ or $b \in \pi_p$, proving that π_p is a prime ideal.

(2) First, let $x \in \pi_p$, so $p \nleq x$. Since $x \wedge p \leq x$, if $p \leq x \wedge p$ held, it would imply $p \leq x$, a contradiction. Therefore, $p \nleq x \wedge p$, which means $x \wedge p \in \pi_p$. Thus $x \in \Gamma_p$, and we have $\pi_p \subseteq \Gamma_p$.

For the reverse inclusion $\Gamma_p \subseteq \pi_p$, assume $y \in \Gamma_p$. By definition, $y \wedge p \in \pi_p$, so $p \nleq y \wedge p$. Note that if $p \leq y$, then $p \leq y \wedge p$ (since $p \leq p$ is always true); thus, by contraposition, $p \nleq y \wedge p$ implies $p \nleq y$. Consequently, $y \in \pi_p$, establishing the equality $\pi_p = \Gamma_p$. □

Corollary 3.20. Let p and q be two cs-prime elements of a frame $\mathcal{L}$. Then the following statements are equivalent:

(1) Either $\pi_p \subseteq \pi_q$ or $\pi_q \subseteq \pi_p$;

(2) Either $p \leq q$ or $q \leq p$.

Proof . (1) $\Rightarrow$ (2) Suppose that $\pi_p \subseteq \pi_q$. If $p \nleq q$, then $q \in \pi_p$ (by definition, $q \nleq q$ is false, so q cannot be in π_q, but the assumption $p \nleq q$ forces $q \in \pi_p$). Since $\pi_p \subseteq \pi_q$, this implies $q \in \pi_q$, which means $q \nleq q$, a clear contradiction. Therefore, we must have $p \leq q$. By symmetry, $\pi_q \subseteq \pi_p$ implies $q \leq p$.

(2) $\Rightarrow$ (1) Assume $p \leq q$ and let $x \in \pi_p$, so $p \nleq x$. If $x \notin \pi_q$, then $q \leq x$, which implies $p \leq q \leq x$, a contradiction. Hence $x \in \pi_q$, so $\pi_p \subseteq \pi_q$. Similarly, if $q \leq p$, then $\pi_q \subseteq \pi_p$. □

4 CS-prime elements under some constructions

We begin this section with the investigation of the stability of cs-prime elements in various frame-theoretic constructions.

Proposition 4.1. Let $\nu : \mathcal{L} \to \mathcal{L}'$ be a frame homomorphism. Then the following statements hold:

(1) If ν is surjective and order-reflecting, then for every cs-prime element p of $\mathcal{L}$, $\nu(p)$ is a cs-prime element of $\mathcal{L}'$.
(2) If ν is order-reflecting and $\nu(p)$ is a cs-prime element of $\mathcal{L}'$, then p is a cs-prime element of $\mathcal{L}$.

Proof . (1) Suppose that p is a cs-prime element of $\mathcal{L}$. Since ν is order-reflecting and $\nu(0) = 0$, we have $\nu(p) \neq 0$.

Let $S' \subseteq \mathcal{L}'$ and suppose that $\nu(p) \leq \bigvee S'$. By the surjectivity of ν, for every $s' \in S'$, there exists $s_{s'} \in \mathcal{L}$ such that $\nu(s_{s'}) = s'$. Put

$$S = \{s_{s'} : s' \in S'\}.$$

Since ν preserves arbitrary joins, we obtain

$$\nu\left(\bigvee S\right) = \bigvee_{s\in S} \nu(s) = \bigvee S'.$$

It follows that $\nu(p) \leq \nu\left(\bigvee S\right)$. Since ν reflects the order, we get $p \leq \bigvee S$. Now, as p is cs-prime, there exists $s \in S$ such that $p \leq s$. Therefore,

$$\nu(p) \leq \nu(s) \in S'.$$

Hence, $\nu(p)$ is a cs-prime element of $\mathcal{L}'$.

(2) Suppose that $\nu(p)$ is a cs-prime element of $\mathcal{L}'$. In particular, $\nu(p) \neq 0$. Let $S \subseteq \mathcal{L}$ and suppose that $p \leq \bigvee S$. Since ν is a frame homomorphism, we have

$$\nu(p) \leq \nu\left(\bigvee S\right) = \bigvee_{s\in S} \nu(s).$$

By the cs-primality of $\nu(p)$, there exists $s \in S$ such that $\nu(p) \leq \nu(s)$. Since ν reflects the order, it follows that $p \leq s$. Moreover, $p \neq 0$, because $\nu(p) \neq 0$. Thus, p is a cs-prime element of $\mathcal{L}$. □

Remark 4.2. The order-reflecting condition in Proposition 4.1 is essential. Indeed, a frame homomorphism preserves the order, but it need not reflect the order. Thus, in general, the implication

$$v(x) \leq v(y) \quad \Longrightarrow \quad x \leq y$$

does not hold.

To see that surjectivity alone is not sufficient, consider the frame

$$\mathcal{L} = \mathcal{P}(\{1, 2\})$$

and the two-element frame

$$\mathcal{L}' = \{0, 1\}.$$

Define $v : \mathcal{L} \to \mathcal{L}'$ by

$$v(A) = \begin{cases} 1, & 1 \in A, \\ 0, & 1 \notin A. \end{cases}$$

Then v is a surjective frame homomorphism.

However, the element $p = \{2\}$ is cs-prime in $\mathcal{L}$, whereas $v(p) = 0$ is not cs-prime in $\mathcal{L}'$. Thus, assertion (1) of the original Proposition 4.1 is false without the order-reflecting assumption.

Similarly, let $p = \{1, 2\}$. Then p is not cs-prime in $\mathcal{L}$, since

$$p = \{1\} \vee \{2\},$$

but p is not below either $\{1\}$ or $\{2\}$. On the other hand, $v(p) = 1$ is cs-prime in $\mathcal{L}'$. Hence, assertion (2) of the original Proposition 4.1 is also false.

In particular, if v is both surjective and order-reflecting, then it is an order isomorphism. Consequently, Proposition 4.1 shows that cs-primality is preserved and reflected by such frame homomorphisms.

We now turn to the stability of cs-prime elements under quotients of frames. Let I be an ideal of a frame $\mathcal{L}$. The quotient frame $\mathcal{L}/I$ is determined by the congruence $\equiv_I$ defined by

$$x \equiv_I y \quad \Longleftrightarrow \quad x \vee i = y \vee i \quad \text{for some } i \in I.$$

We denote the equivalence class of x by $x \vee I$, and write

$$\mathcal{L}/I = \{x \vee I : x \in \mathcal{L}\}.$$

The order on $\mathcal{L}/I$ is given by

$$x \vee I \leq_Q y \vee I \quad \Longleftrightarrow \quad x \leq y \vee i \quad \text{for some } i \in I.$$

With respect to this order, finite meets and arbitrary joins are given by

$$(x \vee I) \wedge_Q (y \vee I) = (x \wedge y) \vee I$$

and

$$\bigvee_{\lambda \in \Lambda} (x_\lambda \vee I) = \left(\bigvee_{\lambda \in \Lambda} x_\lambda \right) \vee I.$$

Consequently, $\mathcal{L}/I$ is again a frame. Moreover, the ideals of $\mathcal{L}$ containing I correspond naturally to the ideals of $\mathcal{L}/I$, and this correspondence restricts to a correspondence between prime ideals. These standard facts about congruences, ideals, and quotient frames can be found, for example, in [6, 7], and we shall use them freely below. In particular, the least element of $\mathcal{L}/I$ is

$$0 \vee I = I.$$

Proposition 4.3. Suppose that p is a cs-prime element of a frame $\mathcal{L}$, and let I be an ideal of $\mathcal{L}$ such that $p \notin I$. Then $p \vee I$ is a cs-prime element of $\mathcal{L}/I$.

Proof . Since $p \notin I$, we have $p \vee I \neq I$, and hence $p \vee I$ is a nonzero element of $\mathcal{L}/I$. Let

$$p \vee I \leq_Q \bigvee_{\lambda \in \Lambda} (s_\lambda \vee I),$$

where $s_\lambda \in \mathcal{L}$ for every $\lambda \in \Lambda$. Since arbitrary joins in $\mathcal{L}/I$ are given by

$$\bigvee_{\lambda \in \Lambda} (s_\lambda \vee I) = \left(\bigvee_{\lambda \in \Lambda} s_\lambda \right) \vee I,$$

let us write $s = \bigvee_{\lambda \in \Lambda} s_\lambda$. By the definition of the order on $\mathcal{L}/I$, there exists $i \in I$ such that

$$p \leq s \vee i = \left(\bigvee_{\lambda \in \Lambda} s_\lambda \right) \vee i = \bigvee_{\lambda \in \Lambda} (s_\lambda \vee i).$$

Since p is cs-prime in $\mathcal{L}$, it follows that either

$$p \leq s_\lambda \quad \text{for some } \lambda \in \Lambda,$$

or

$$p \leq i.$$

The latter is impossible, because $i \in I$ and I is a lower set (ideal); indeed, $p \leq i \in I$ would imply $p \in I$, which contradicts the assumption that $p \notin I$. Hence, we must have

$$p \leq s_\lambda \quad \text{for some } \lambda \in \Lambda.$$

Consequently,

$$p \vee I \leq_Q s_\lambda \vee I.$$

Thus, $p \vee I$ is a cs-prime element in $\mathcal{L}/I$. □

Proposition 4.4. If p is a cs-prime element of a frame $\mathcal{L}$, then $p \vee \pi_p$ is an atom in $\mathcal{L}/\pi_p$.

Proof . Recall that

$$\pi_p = \{x \in \mathcal{L} : p \nleq x\}.$$

By Theorem 3.9, π_p is an ideal of $\mathcal{L}$, so the quotient frame $\mathcal{L}/\pi_p$ is defined, with least element

$$0_{\mathcal{L}/\pi_p} = \pi_p = 0 \vee \pi_p.$$

We must show that $p \vee \pi_p$ is an atom of $\mathcal{L}/\pi_p$, that is, there is no element $b \vee \pi_p \in \mathcal{L}/\pi_p$ such that

$$\pi_p \lneqq b \vee \pi_p \lneqq p \vee \pi_p.$$

Suppose, for a contradiction, that such an element exists. In particular, $b \vee \pi_p \neq p \vee \pi_p$, so $b \not\equiv_{\pi_p} p$; equivalently,

$$p \nleq b \vee i \qquad \text{for every } i \in \pi_p.$$

Taking $i = 0$ (which lies in π_p, since π_p is an ideal and hence contains 0), we obtain in particular that

$$p \not\leq b.$$

By the definition of π_p, this means precisely that $b \in \pi_p$. But then

$$b \vee \pi_p = \pi_p = 0_{\mathcal{L}/\pi_p},$$

contradicting the assumption that $\pi_p \lneqq b \vee \pi_p$. Hence, no element $b \vee \pi_p$ lies strictly between π_p and $p \vee \pi_p$.

Finally, $p \vee \pi_p \neq \pi_p$: indeed, if $p \vee \pi_p = \pi_p$, then $p \leq p \vee \pi_p = \pi_p$, which means $p \in \pi_p$, contradicting the definition of π_p (since $p \not\leq p$ is false). Thus, $p \vee \pi_p$ is a nonzero element of $\mathcal{L}/\pi_p$ with no element strictly below it other than $0_{\mathcal{L}/\pi_p}$. Therefore, $p \vee \pi_p$ is an atom of $\mathcal{L}/\pi_p$. □

Proposition 4.5. Suppose that p is a cs-prime element of a frame $\mathcal{L}$ and let I be an ideal of $\mathcal{L}$ such that $p \notin I$. Then $\pi_{p\vee I}$ is a prime ideal of $\mathcal{L}/I$.

Proof . We divide the argument into three steps.

Step 1: $I \subseteq \pi_p$**.** Let $x \in I$. Suppose, for a contradiction, that $x \notin \pi_p$. By the definition of $\pi_p = \{y \in \mathcal{L} : p \not\leq y\}$, this means $p \leq x$. Since I is an ideal and $x \in I$, the element $p = p \wedge x$ also belongs to I, contradicting the hypothesis that $p \notin I$. Hence, $x \in \pi_p$, and therefore $I \subseteq \pi_p$.

Step 2: $\pi_{p\vee I} = \pi_p/I$**.** Since $I \subseteq \pi_p$, the set

$$\pi_p/I = \{x \vee I \in \mathcal{L}/I : x \in \pi_p\}$$

is a well-defined subset of $\mathcal{L}/I$.

First, let $x \vee I \in \pi_{p\vee I}$. By definition,

$$p \vee I \not\leq_Q x \vee I,$$

where $\leq_Q$ denotes the order in $\mathcal{L}/I$. Recall that for elements $a, b \in \mathcal{L}$,

$$a \vee I \leq_Q b \vee I \quad \Longleftrightarrow \quad a \leq b \vee i \quad \text{for some } i \in I.$$

Thus, $p \vee I \not\leq_Q x \vee I$ means precisely that

$$p \not\leq x \vee i \qquad \text{for every } i \in I.$$

Taking $i = 0 \in I$, we obtain in particular $p \not\leq x$, that is, $x \in \pi_p$. Hence, $x \vee I \in \pi_p/I$, and so

$$\pi_{p\vee I} \subseteq \pi_p/I.$$

Conversely, let $x \vee I \in \pi_p/I$, so $x \in \pi_p$ (i.e., $p \not\leq x$). We must show that $p \vee I \not\leq_Q x \vee I$, meaning that $p \not\leq x \vee i$ for every $i \in I$. Suppose, on the contrary, that $p \leq x \vee i$ for some $i \in I$. Since p is cs-prime, Definition 3.1(2) applied to the set $S = \{x, i\}$ yields

$$p \leq x \quad \text{or} \quad p \leq i.$$

The first alternative contradicts $x \in \pi_p$, and the second contradicts $p \notin I$, because $i \in I$ and I is a lower set. Both alternatives are therefore impossible, so $p \not\leq x \vee i$ for every $i \in I$. Hence, $p \vee I \not\leq_Q x \vee I$, which implies that $x \vee I \in \pi_{p\vee I}$. This proves that

$$\pi_p/I \subseteq \pi_{p\vee I}.$$

Combining the two inclusions gives $\pi_{p\vee I} = \pi_p/I$.

Step 3: $\pi_{p\vee I}$ is a prime ideal of $\mathcal{L}/I$. Since p is cs-prime, Theorem 3.19 asserts that π_p is a prime ideal of $\mathcal{L}$. By the standard correspondence between ideals of $\mathcal{L}$ containing I and ideals of the quotient frame $\mathcal{L}/I$ (which restricts to a correspondence between prime ideals), the ideal π_p/I of $\mathcal{L}/I$ is prime. By Step 2, $\pi_{p\vee I} = \pi_p/I$, and thus $\pi_{p\vee I}$ is a prime ideal of $\mathcal{L}/I$. □

Lemma 4.6. Let L be a frame and let $p \in L$. Define

$$\tau_p = \{x \in L : x \leq p\},$$

$$\chi_p = \{x \in L : p \not\leq x\},$$

and

$$\nu_p = \{x \in L : x \wedge p \in \chi_p\}.$$

Then the following statements hold:

(1) τ_p is an ideal of L.
(2) If $p \leq q$, then $\tau_p \subseteq \tau_q$.
(3) If p is cs-prime, then χ_p is an ideal of L.

(4) If p is cs-prime, then ν_p is an ideal of L.

Proof . We use Lemma 2.1 throughout: a nonempty subset J of L is an ideal if and only if $x \vee y \in J$ and $x \wedge z \in J$ for all $x, y \in J$ and $z \in L$.

(1) Clearly, $0 \in \tau_p$, since $0 \leq p$. Let $x, y \in \tau_p$ and $z \in L$. Then $x \leq p$ and $y \leq p$, so $x \vee y \leq p$. Also, $x \wedge z \leq x \leq p$. Thus $x \vee y \in \tau_p$ and $x \wedge z \in \tau_p$. By Lemma 2.1, τ_p is an ideal of L.

(2) Let $x \in \tau_p$. Then $x \leq p$. Since $p \leq q$, we obtain $x \leq q$, and hence $x \in \tau_q$. Therefore,

$$\tau_p \subseteq \tau_q.$$

(3) Assume that p is cs-prime. Since $p \neq 0$, we have $0 \in \chi_p$. Let $a, b \in \chi_p$ and $c \in L$. Thus

$$p \nleq a \qquad \text{and} \qquad p \nleq b.$$

First, since $a \wedge c \leq a$, we cannot have $p \leq a \wedge c$. Hence $a \wedge c \in \chi_p$.

Next, suppose that $a \vee b \notin \chi_p$. Then $p \leq a \vee b$. Since p is cs-prime, there exists an element of $\{a, b\}$ above p. Thus $p \leq a$ or $p \leq b$, a contradiction. Consequently, $a \vee b \in \chi_p$. By Lemma 2.1, χ_p is an ideal of L.

(4) By the definitions of ν_p and χ_p, for every $x \in L$ we have

$$\begin{aligned} x \in \nu_p &\iff x \wedge p \in \chi_p \\ &\iff p \nleq x \wedge p \\ &\iff p \nleq x \\ &\iff x \in \chi_p. \end{aligned}$$

Therefore, $\nu_p = \chi_p$. By (3), χ_p is an ideal of L. Hence ν_p is also an ideal of L. □

Proposition 4.7. Suppose that p is a cs-prime element of a frame $\mathcal{L}$ and let I be an ideal of $\mathcal{L}$ such that $p \notin I$. Then the following statements are true:

(1) $I \subseteq \chi_p$;
(2) $\chi_{p \vee I} = \chi_p / I$;
(3) $\nu_{p \vee I} = \nu_p / I$.

Proof . Recall the definitions

$$\tau_x = \{y \in \mathcal{L} : y \leq x\}, \quad \chi_p = \{x \in \mathcal{L} : p \notin \tau_x\} = \{x \in \mathcal{L} : p \not\leq x\},$$

and

$$\nu_p = \{x \in \mathcal{L} : x \wedge p \in \chi_p\}.$$

By Lemma 4.6, both χ_p and ν_p are ideals of $\mathcal{L}$ (the former because p is cs-prime).

(1) Let $x \in I$. Suppose, for a contradiction, that $x \notin \chi_p$, i.e., $p \in \tau_x$, so $p \leq x$. Since I is an ideal, $x \in I$ and $p \leq x$ imply that $p = p \wedge x \in I$, contradicting the hypothesis that $p \notin I$. Hence, $x \in \chi_p$, and therefore $I \subseteq \chi_p$.

(2) Since $I \subseteq \chi_p$ by (1), the set

$$\chi_p/I = \{x \vee I \in \mathcal{L}/I : x \in \chi_p\}$$

is a well-defined subset of $\mathcal{L}/I$. Here, for $x \in \mathcal{L}$,

$$x \vee I \in \chi_{p\vee I} \iff p \vee I \notin \tau_{x\vee I} \iff p \vee I \not\leq_Q x \vee I.$$

Recall that on $\mathcal{L}/I$,

$$a \vee I \leq_Q b \vee I \iff a \leq b \vee i \quad \text{for some } i \in I.$$

Thus, $p \vee I \not\leq_Q x \vee I$ means

$$p \not\leq x \vee i \qquad \text{for every } i \in I. \qquad (*)$$

First inclusion ($\chi_{p\vee I} \subseteq \chi_p/I$): Let $x \vee I \in \chi_{p\vee I}$. By $(*)$ with $i = 0 \in I$, we get $p \not\leq x$, i.e., $x \in \chi_p$. Hence, $x \vee I \in \chi_p/I$.

Second inclusion ($\chi_p/I \subseteq \chi_{p\vee I}$): Let $x \vee I \in \chi_p/I$, so $x \in \chi_p$, i.e., $p \not\leq x$. We must show that $p \vee I \not\leq_Q x \vee I$, meaning that $p \not\leq x \vee i$ for every $i \in I$. Suppose, on the contrary, that $p \leq x \vee i$ for some $i \in I$. Since p is cs-prime, Definition 3.1(2) applied to the set $S = \{x, i\}$ gives

$$p \leq x \quad \text{or} \quad p \leq i.$$

The first alternative contradicts $x \in \chi_p$. The second contradicts $p \notin I$, because $i \in I$ and I is a lower set. Hence, no such i exists, so $p \not\leq x \vee i$ for every $i \in I$, and thus $x \vee I \in \chi_{p\vee I}$. Therefore, $\chi_{p\vee I} = \chi_p/I$.

(3) First, we show that $I \subseteq \nu_p$. Let $y \in I$. By (1), $y \in \chi_p$. Since χ_p is an ideal by Lemma 4.6(3) and $y \in \chi_p$, we have

$$y \wedge p \leq y \in \chi_p.$$

As χ_p is downward closed (being an ideal), $y \wedge p \in \chi_p$, that is, $y \in \nu_p$. Hence, $I \subseteq \nu_p$, and therefore $\nu_p/I = \{x \vee I : x \in \nu_p\}$ is a well-defined subset of $\mathcal{L}/I$.

Now, using the definition of ν, the fact that meets in $\mathcal{L}/I$ are computed componentwise, and (2), we compute

$$\begin{aligned}
\nu_{p\vee I} &= \{x \vee I \in \mathcal{L}/I : (p \vee I) \wedge_Q (x \vee I) \in \chi_{p\vee I}\} \\
&= \{x \vee I \in \mathcal{L}/I : (p \wedge x) \vee I \in \chi_{p\vee I}\} \\
&= \{x \vee I \in \mathcal{L}/I : (p \wedge x) \vee I \in \chi_p/I\} \\
&= \{x \vee I \in \mathcal{L}/I : p \wedge x \in \chi_p\} \\
&= \{x \vee I \in \mathcal{L}/I : x \in \nu_p\} \\
&= \nu_p/I.
\end{aligned}$$

This proves (3). □

Definition 4.8. A frame L is called a $\wedge$-factorization frame if every nonzero element of L can be expressed as a finite meet of cs-prime elements; that is, for every $x \in L \setminus \{0\}$, there exist an integer $n \geq 1$ and cs-prime elements $p_1, \ldots, p_n \in L$ such that

$$x = \bigwedge_{i=1}^{n} p_i.$$

Proposition 4.9. Let I be a proper ideal of a frame $\mathcal{L}$. If $\mathcal{L}$ is a $\wedge$-factorization frame, then $\mathcal{L}/I$ is a $\wedge$-factorization frame.

Proof . Suppose that $\mathcal{L}$ is a $\wedge$-factorization frame and let $p \vee I$ be a proper element of $\mathcal{L}/I$ (so $p \notin I$). Let $p = \bigwedge_{i=1}^{n} p_i$ be a $\wedge$-factorization of p, which implies that $p_i \notin I$ for each $i \in \{1, 2, \ldots, n\}$, as I is an ideal. Then

$$p \vee I = \left(\bigwedge_{i=1}^{n} p_i\right) \vee I = (p_1 \vee I) \wedge_Q \cdots \wedge_Q (p_n \vee I).$$

By Proposition 4.3, we infer that $p_i \vee I$ is a cs-prime element of $\mathcal{L}/I$ for each $i \in \{1, 2, \ldots, n\}$. This means that $\mathcal{L}/I$ is a $\wedge$-factorization frame. □

We now turn to product frames. Recall that if $(\mathcal{L}_1, \leq_1)$ and $(\mathcal{L}_2, \leq_2)$ are frames, then $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$, ordered componentwise by $\leq_c$, is again a frame. For $S \subseteq \mathcal{L}$, writing $S_1 = \{x_1 \in \mathcal{L}_1 : (x_1, y_2) \in S \text{ for some } y_2\}$ and $S_2 = \{y_2 \in \mathcal{L}_2 : (x_1, y_2) \in S \text{ for some } x_1\}$, one has $S \subseteq S_1 \times S_2$ and $\bigvee S = (\bigvee S_1, \bigvee S_2)$; this is a standard fact about products of frames (see, e.g., [10, 11]).

Proposition 4.10. Suppose that $L = L_1 \times L_2$ is a product frame, and let $p_i \in L_i$ for $i \in \{1, 2\}$. If (p_1, p_2) is a cs-prime element of L, then each p_i is either zero or a cs-prime element of L_i.

Proof . Suppose that (p_1, p_2) is a cs-prime element of $L = L_1 \times L_2$.

We first show that p_1 satisfies the cs-primality condition whenever it is nonzero.

Let $S_1 \subseteq L_1$ and suppose that $p_1 \leq_1 \bigvee S_1$. Set

$$T = \{(s_1, 1_{L_2}) : s_1 \in S_1\}.$$

Since joins in a product frame are computed componentwise, we have

$$\bigvee T = \left(\bigvee S_1, 1_{L_2}\right).$$

Consequently, $(p_1, p_2) \leq_c \bigvee T$. By the cs-primality of (p_1, p_2), there exists $s_1 \in S_1$ such that $(p_1, p_2) \leq_c (s_1, 1_{L_2})$. In particular, $p_1 \leq_1 s_1$.

Thus, if $p_1 \neq 0$, then p_1 is a cs-prime element of L_1.

The proof for p_2 is analogous. Hence, each p_i is either zero or a cs-prime element of L_i. □

Example 4.11. The converse of Proposition 4.10 does not hold in general. Let

$$\mathcal{L}_1 = \mathcal{L}_2 = \{0, 1\}$$

be the two-element chain, and let $p_1 = p_2 = 1$. The element 1 is cs-prime in each of $\mathcal{L}_1$ and $\mathcal{L}_2$.

Consider the product frame $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$. Then $(p_1, p_2) = (1, 1)$. Let

$$S = \{(1, 0), (0, 1)\}.$$

We have

$$\bigvee S = (1, 0) \vee_c (0, 1) = (1, 1),$$

so $(1,1) \leq_c \bigvee S$. However,

$$(1,1) \not\leq_c (1,0) \qquad \text{and} \qquad (1,1) \not\leq_c (0,1).$$

Thus $(1,1)$ is not cs-prime in $\mathcal{L}$. Therefore, the converse of Proposition 4.10 does not hold in general.

The following result provides a condition under which the converse of Proposition 4.10 is true.

Theorem 4.12. Suppose that $L = L_1 \times L_2$ is a product frame and let $p_i \in L_i$ for $i \in \{1,2\}$. Then the following statements hold:

(1) $(p_1, 0)$ is a cs-prime element of L if and only if p_1 is a cs-prime element of L_1.
(2) $(0, p_2)$ is a cs-prime element of L if and only if p_2 is a cs-prime element of L_2.

Proof . (1) Suppose first that $(p_1, 0)$ is a cs-prime element of L. In particular, $p_1 \neq 0$.

Let $S_1 \subseteq L_1$ and suppose that $p_1 \leq_1 \bigvee S_1$. Consider the subset

$$T = \{(s_1, 1_{L_2}) : s_1 \in S_1\}$$

of L. Since joins in a product frame are computed componentwise, we have

$$\bigvee T = \left(\bigvee S_1, 1_{L_2}\right).$$

It follows that $(p_1, 0) \leq_c \bigvee T$. By the cs-primality of $(p_1, 0)$, there exists $s_1 \in S_1$ such that

$$(p_1, 0) \leq_c (s_1, 1_{L_2}).$$

In particular, $p_1 \leq_1 s_1$. Thus, p_1 is a cs-prime element of L_1.

Conversely, suppose that p_1 is a cs-prime element of L_1. In particular, $p_1 \neq 0$.

Let $S \subseteq L$ and suppose that $(p_1, 0) \leq_c \bigvee S$. Define

$$S_1 = \{x_1 \in L_1 : (x_1, x_2) \in S \text{ for some } x_2 \in L_2\}$$

and

$$S_2 = \{x_2 \in L_2 : (x_1, x_2) \in S \text{ for some } x_1 \in L_1\}.$$

By the componentwise description of joins in product frames, we have

$$\bigvee S = \left(\bigvee S_1, \bigvee S_2\right).$$

Consequently, $p_1 \leq_1 \bigvee S_1$. Since p_1 is cs-prime in L_1, there exists $s_1 \in S_1$ such that $p_1 \leq_1 s_1$. By the definition of S_1, there exists $s_2 \in L_2$ such that $(s_1, s_2) \in S$. Since $0 \leq_2 s_2$, we obtain $(p_1, 0) \leq_c (s_1, s_2)$. Therefore, $(p_1, 0)$ is a cs-prime element of L.

(2) The proof is analogous to that of (1), with the roles of L_1 and L_2 interchanged.

For completeness, suppose first that $(0, p_2)$ is a cs-prime element of L. In particular, $p_2 \neq 0$.

Let $S_2 \subseteq L_2$ and suppose that $p_2 \leq_2 \bigvee S_2$. Consider

$$U = \{(1_{L_1}, s_2) : s_2 \in S_2\}.$$

Then $\bigvee U = (1_{L_1}, \bigvee S_2)$, and hence $(0, p_2) \leq_c \bigvee U$. By the cs-primality of $(0, p_2)$, there exists $s_2 \in S_2$ such that $(0, p_2) \leq_c (1_{L_1}, s_2)$. Thus, $p_2 \leq_2 s_2$, and p_2 is cs-prime in L_2.

Conversely, suppose that p_2 is a cs-prime element of L_2. In particular, $p_2 \neq 0$.

Let $S \subseteq L$ and suppose that $(0, p_2) \leq_c \bigvee S$. Using the subsets S_1 and S_2 defined above, we have

$$\bigvee S = \left(\bigvee S_1, \bigvee S_2\right).$$

It follows that $p_2 \leq_2 \bigvee S_2$. Since p_2 is cs-prime in L_2, there exists $s_2 \in S_2$ such that $p_2 \leq_2 s_2$. By the definition of S_2, there exists $s_1 \in L_1$ such that $(s_1, s_2) \in S$. Since $0 \leq_1 s_1$, we get $(0, p_2) \leq_c (s_1, s_2)$. Therefore, $(0, p_2)$ is a cs-prime element of L. □

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Conflict of Interest

The author declares that he has no conflict of interest.

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