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A new hybrid computational method based on Legendre wavelets and modified hat functions for solving nonlinear stochastic Itó–volterra integral equations

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Abstract

This paper develops an efficient hybrid method combining Legendre wavelets and modified hat functions to solve nonlinear stochastic Itó–Volterra integral equations. By exploiting the global orthogonality of Legendre wavelets and local quadratic interpolation of modified hat functions, the approach overcomes single-basis limitations. A new stochastic operational matrix for Legendre wavelets is derived from the modified hat function basis, which is exact for polynomial degrees up to two and an $O(h^3)$ approximation otherwise. Its existence, uniqueness, invertibility, and numerical stability are rigorously proven. Mean-square convergence analysis, using Itô isometry and the Burkholder–Davis–Gundy inequality, confirms a third-order rate. Four numerical examples, including a self-convergence test, demonstrate high accuracy and efficiency. Comparisons with Wilson wavelet, Legendre wavelet, hat function, and Monte Carlo methods confirm 10^{-10} mean-square errors with lower cost for $\hat{m} \geq 96$.

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1 Introduction

Stochastic modelling has become an indispensable tool across a wide range of scientific disciplines, including geophysics, fluid mechanics, chemistry, biology, epidemiology, microelectronics, theoretical physics, economics, and finance [1, 2]. In these fields, the dynamics of many systems are driven by Gaussian white noise, and their mathematical description naturally leads to stochastic differential equations and, in more complex settings, to stochastic integral and integro-differential equations.

Numerical methods for stochastic integral equations have received considerable attention over the past two decades [3, 4, 5]. Among the most widely used approaches are those based on orthogonal polynomials and wavelets, including hat functions [6, 17], Chebyshev polynomials [7], hybrid Legendre block-pulse functions [8], Haar wavelets [9], and Legendre wavelets [10, 11, 12].

Recent developments include spectral collocation methods combined with Monte Carlo sampling [21], two-dimensional wavelet techniques [22], mean-square stability analysis of split-step theta methods [19], wavelet collocation for fractional stochastic equations [18], fast θ-Maruyama schemes for stochastic Volterra equations [20], and mean-square stability of numerical methods for stochastic problems [23].

1.1 Research Gap and Motivation

Despite these advances, a critical examination of the existing literature reveals five fundamental limitations that motivate the present work.

Limitation 1: Global versus local bases. Methods based on global orthogonal bases—Chebyshev, Legendre, and Bernstein polynomials—achieve spectral accuracy in the interior of the domain. However, they suffer from two well-known drawbacks: (i) their convergence deteriorates near the endpoints (Gibbs-type oscillations), and (ii) they require many basis functions to resolve localized features. Conversely, purely local bases—hat functions, Haar wavelets, and block-pulse functions—provide excellent local approximation but lack the spectral accuracy of global bases, requiring very fine discretizations that increase both the algebraic system dimension and the computational cost.

Limitation 2: Linearization errors in nonlinear problems. Many existing wavelet-based approaches linearize the nonlinear terms ψ_1 and ψ_2 before discretization, introducing an additional approximation error that is difficult to control and often dominates the discretization error [8].

Limitation 3: Lack of rigorous stochastic operational matrices. The derivation of stochastic operational matrices for wavelets is frequently carried out formally, without justification of existence, uniqueness, invertibility, or numerical stability [10]. This is problematic because the stability of the resulting algebraic system depends critically on

the condition number of the transformation between the wavelet basis and the collocation basis.

Limitation 4: Pathwise versus mean-square error analysis. Existing error analyses are often carried out pathwise, requiring deterministic bounds on the Wiener process that do not hold uniformly. Recent work by Li et al. [19] and Wang et al. [20] has established mean-square stability frameworks for stochastic Volterra equations, but their application to wavelet-based methods remains unexplored.

Limitation 5: The specific gap addressed in this work. To our knowledge, no existing method simultaneously (i) combines a globally orthogonal basis with a locally interpolatory basis, (ii) derives the corresponding stochastic operational matrix with a rigorous existence and stability analysis, and (iii) establishes convergence in the mean-square sense with a provable rate. The method proposed in this paper addresses all three issues simultaneously.

1.2 Novelty and Contributions

The principal contributions of this paper are as follows.

(A) Hybrid basis. We combine Legendre wavelets, which provide global orthogonality and spectral accuracy, with modified hat functions, which are interpolatory and provide local quadratic accuracy. The latter are chosen because (i) their coefficients coincide with the function values at the nodes; (ii) they provide local quadratic accuracy; (iii) they yield a well-conditioned invertible transformation matrix $\mathbf{T}$ between the two bases. The transformation is exact when $M \leq 3$ and the modified hat function mesh conforms to the wavelet knots; otherwise, it is an $O(h^3)$ approximation.

(B) New stochastic operational matrix with rigorous analysis. We derive a new stochastic operational matrix P_s for Legendre wavelets by transforming the modified-hat-function-based stochastic operational matrix into the Legendre wavelet basis. Unlike previous works, we rigorously establish existence, uniqueness, invertibility, and a condition-number bound.

(C) Rigorous mean-square convergence. We perform the error analysis entirely in the mean-square norm $L^2(\Omega)$ using the Itô isometry and the Burkholder–Davis–Gundy inequality, followed by a stochastic Gronwall inequality. This yields a rigorous $O(h^3)$ convergence rate and corrects an error in the previously published Legendre wavelet truncation-error theorem [12, Theorem 4.1].

(D) Correction of existing results. We show that the previously published estimate $O(M^{-1})$ should be replaced by $O(h^3)$ in the L^2 norm.

(E) Nonlinear treatment without linearization. The nonlinear functions are handled exactly, without any linearization error.

1.3 Organization of the Paper

Section 2 reviews the necessary preliminaries. Section 3 presents the proposed hybrid method. Section 4 provides a rigorous mean-square convergence analysis. Section 5 reports four numerical experiments. Section 6 concludes the paper.

2 Preliminaries and Notation

2.1 Notation

All vectors and matrices have consistent dimensions, established by the identity

$$N + 1 = \hat{m} = 2^{k-1}M. \tag{2.1}$$

Table 1: Notation and dimensions used throughout the paper.

Symbol	Definition	Dimension
k	Wavelet level ($k \geq 1$)	—
M	Number of Legendre polynomial terms per subinterval	—
ℓ	Translation index, $\ell = 1, \ldots, 2^{k-1}$	—
m	Polynomial degree index, $m = 0, \ldots, M-1$	—
$\hat{m}$	Total number of Legendre wavelets, $\hat{m} = 2^{k-1}M$	—
N	Number of MHF subintervals, $N = \hat{m} - 1$	—
$N + 1$	Number of MHF basis functions, $N + 1 = \hat{m}$	—
T	Final time	—
H	Wavelet subinterval width, $H = T/2^{k-1}$	—
h	MHF mesh width, $h = T/N = T/(\hat{m} - 1)$	—
$\Psi(t)$	Legendre wavelet vector	$\hat{m} \times 1$
$\Phi(t)$	Modified hat function vector	$\hat{m} \times 1$
$\mathbf{T}$	Transformation matrix	$\hat{m} \times \hat{m}$
P	Integration matrix for Legendre wavelets	$\hat{m} \times \hat{m}$
P_s	Stochastic matrix for Legendre wavelets	$\hat{m} \times \hat{m}$
$\hat{P}_s$	Stochastic matrix for modified hat functions	$\hat{m} \times \hat{m}$

The method is developed on the general interval $[0, T]$ with $T > 0$. The Legendre wavelets are defined on the unit interval $[0, 1]$ via the standard affine change of variables $t \mapsto t/T$; all formulas extend to $[0, T]$ by this mapping. In the numerical experiments we take $T = 1$ for simplicity.

2.2 The Wiener Process

Definition 2.1 (Wiener process). A stochastic process $\{\mathcal{W}(t)\}_{t\geq 0}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Wiener process if:

1. $\mathcal{W}(0) = 0$ almost surely;
2. $t \mapsto \mathcal{W}(t)$ is continuous almost surely;
3. $\mathcal{W}(t) - \mathcal{W}(s) \sim \mathcal{N}(0, t-s)$ for $0 \le s < t$;
4. increments on disjoint intervals are independent.

Remark 2.2. Definition 2.1 is the standard characterization of the Wiener process . We emphasize that no uniform deterministic bound of the form $\sup_{t\in[0,T]} |\mathcal{W}(t)| \le C$ holds almost surely; consequently, all error estimates in Section 4 are formulated in the mean-square sense.

2.3 Mean-Square Function Spaces

For a stochastic process $X : [0,T] \times \Omega \to \mathbb{R}$, we define the mean-square norm

$$\|X\|_{L^2(\Omega)} := \left(\mathbb{E}\left[\sup_{0\le t\le T} |X(t)|^2\right]\right)^{1/2}. \tag{2.2}$$

For deterministic functions, we use the supremum norm

$$\|f\|_{\infty,D} := \sup_{t\in D} |f(t)|. \tag{2.3}$$

Lemma 2.3 (Itô isometry). For any $G \in L^2(\Omega \times [0,T])$:

$$\mathbb{E}\left[\left|\int_0^t G(s)\,\mathrm{d}\mathcal{W}(s)\right|^2\right] = \mathbb{E}\left[\int_0^t |G(s)|^2\,\mathrm{d}s\right]. \tag{2.4}$$

Lemma 2.4 (Burkholder–Davis–Gundy inequality). For $p = 2$, with $C_{\mathrm{BDG}} \le 4$ [**?**]:

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t G(s)\,\mathrm{d}\mathcal{W}(s)\right|^2\right] \le C_{\mathrm{BDG}}^2\,\mathbb{E}\left[\int_0^T |G(s)|^2\,\mathrm{d}s\right]. \tag{2.5}$$

Lemma 2.5 (Stochastic Gronwall inequality). Let $u : [0,T] \to [0,\infty)$ be a nonnegative measurable process satisfying, for some constants $\alpha, \beta \ge 0$:

$$\mathbb{E}\left[\sup_{0\le r\le t} u(r)\right] \le \alpha + \beta \int_0^t \mathbb{E}\left[\sup_{0\le r\le s} u(r)\right]\mathrm{d}s. \tag{2.6}$$

Then $\mathbb{E}[\sup_{0\le t\le T} u(t)] \le \alpha e^{\beta T}$.

2.4 Legendre Wavelets

Let $k \geq 1$ be the wavelet level, $M \geq 1$ the number of Legendre polynomial terms, and P_m the Legendre polynomial of degree m on $[-1, 1]$. The Legendre wavelets on $[0, 1]$ are [11, 21]

$$\psi_{\ell m}(t) = \begin{cases} \sqrt{m + \frac{1}{2}}\, 2^{k/2} P_m(2^k t - 2\ell + 1), & \frac{\ell - 1}{2^{k-1}} \leq t < \frac{\ell}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.7}$$

with $\ell = 1, \ldots, 2^{k-1}$ and $m = 0, \ldots, M - 1$. The family $\{\psi_{\ell m}\}$ is orthonormal on $L^2[0, 1]$, and the total number of wavelets is $\hat{m} = 2^{k-1} M$.

Any $f \in L^2[0, 1]$ can be expanded as

$$f(t) \approx \sum_{\ell=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{\ell m} \psi_{\ell m}(t) = C^T \Psi(t), \tag{2.8}$$

where $C = [c_{11}, \ldots, c_{2^{k-1}, M-1}]^T$ and $\Psi(t) = [\psi_{11}(t), \ldots, \psi_{2^{k-1}, M-1}(t)]^T$ are both $\hat{m} \times 1$ vectors.

2.5 Modified Hat Functions

Let $N \geq 2$ be a positive integer and $h = T/N$. The modified hat functions $\{h_i\}_{i=0}^{N}$ on $[0, T]$ are defined as follows [6, 17]. The boundary functions are

$$h_0(t) = \begin{cases} \dfrac{1}{2h^2}(t - h)(t - 2h), & 0 \leq t \leq 2h, \\ 0, & \text{otherwise}, \end{cases} \tag{2.9}$$

$$h_N(t) = \begin{cases} \dfrac{1}{2h^2}(t - (T - h))(t - (T - 2h)), & T - 2h \leq t \leq T, \\ 0, & \text{otherwise}. \end{cases} \tag{2.10}$$

For odd i with $1 \leq i \leq N - 1$:

$$h_i(t) = \begin{cases} -\dfrac{1}{h^2}(t - (i - 1)h)(t - (i + 1)h), & (i - 1)h \leq t \leq (i + 1)h, \\ 0, & \text{otherwise}. \end{cases} \tag{2.11}$$

For even i with $2 \le i \le N-2$:

$$h_i(t) = \begin{cases} \dfrac{1}{2h^2}(t-(i-1)h)(t-(i-2)h), & (i-2)h \le t \le ih, \\ \dfrac{1}{2h^2}(t-(i+1)h)(t-(i+2)h), & ih \le t \le (i+2)h, \\ 0, & \text{otherwise.} \end{cases} \tag{2.12}$$

These functions satisfy the interpolation property [13, 17]:

$$h_i(jh) = \delta_{ij}, \qquad i, j = 0, 1, \ldots, N. \tag{2.13}$$

Any continuous X on $[0, T]$ can be approximated as

$$X(t) \approx \sum_{i=0}^{N} X(ih) h_i(t) = X^T \Phi(t), \tag{2.14}$$

where $X = [X(0), X(h), \ldots, X(Nh)]^T$ and $\Phi(t) = [h_0(t), \ldots, h_N(t)]^T$, both of dimension $N + 1 = \hat{m}$.

3 The Proposed Hybrid Method

3.1 Transformation Between Legendre Wavelets and Modified Hat Functions

The key observation is that the modified hat function coefficients in (2.14) are exactly the function values at the nodes $t_i = ih$. Applying this to each Legendre wavelet component ψ_j gives

$$\psi_j(t) = \sum_{i=0}^{N} \psi_j(t_i) h_i(t), \qquad j = 1, \ldots, \hat{m}, \tag{3.1}$$

which is exact when each ψ_j is a polynomial of degree at most 2 on every MHF subinterval (i.e., $M \le 3$) and the MHF mesh conforms to the wavelet knots. Otherwise, the identity holds up to the MHF interpolation error $O(h^3)$ established in Theorem 4.4.

In matrix form:

$$\Psi(t) = \mathbf{T}\Phi(t), \qquad \Phi(t) = \mathbf{T}^{-1}\Psi(t), \tag{3.2}$$

with the transformation matrix

$$\mathbf{T}_{\hat{m}\times\hat{m}} = [\Psi(t_0), \Psi(t_1), \ldots, \Psi(t_N)] \in \mathbb{R}^{\hat{m}\times\hat{m}}. \tag{3.3}$$

Proposition 3.1. The matrix $\mathbf{T}$ is invertible, and the transformation (3.2) is exact for all $t \in [0, T]$ under the conditions stated above.

Proof . Legendre wavelets are linearly independent on $[0, T]$. The interpolation nodes $\{t_i\}_{i=0}^{N}$ are distinct, and the MHF interpolation is unisolvent (the collocation matrix $[\psi_j(t_i)]$ is nonsingular). Hence $\mathbf{T}$ is invertible. When $M \leq 3$, each Legendre wavelet restricted to an MHF subinterval is a polynomial of degree at most 2; identity (3.1) is then exact because both sides are polynomials of degree ≤ 2 on each subinterval agreeing at the three nodes. For $M \geq 4$, the interpolation error is bounded by Theorem 4.4. □

Remark 3.2 (Consistency of dimensions). Both $\Psi(t)$ and $\Phi(t)$ are $\hat{m} \times 1$ vectors, and $\mathbf{T}$ is $\hat{m} \times \hat{m}$. The transformation (3.2) is therefore dimensionally consistent: $\mathbf{T}\Phi(t)$ is $\hat{m} \times 1$, matching the dimension of $\Psi(t)$.

Remark 3.3 (Parameter selection). We choose $\hat{m} = 2^{k-1}M$ and set $N = \hat{m}-1$. We select $M \in \{2, 3\}$ so that the exact transformation holds. Examples: $\hat{m} = 32$ $(k = 5, M = 2)$, $\hat{m} = 48$ $(k = 5, M = 3)$, $\hat{m} = 64$ $(k = 6, M = 2)$, $\hat{m} = 96$ $(k = 6, M = 3)$, $\hat{m} = 128$ $(k = 7, M = 2)$.

3.2 Operational Matrices

The integration operational matrix P for Legendre wavelets is

$$\int_0^t \Psi(s)\,\mathrm{d}s \approx P\Psi(t), \qquad P \in \mathbb{R}^{\hat{m}\times\hat{m}}, \tag{3.4}$$

given explicitly in [11]. Its block structure is $P = (P_{ij})$ with $P_{ij} = A_i$ for $i = j$ and $P_{ij} = 0$ for $i \neq j$, where A_i is the $M \times M$ block associated with the i-th wavelet level. The leading 3×3 sub-block of A_i is

$$A_i^{(3\times3)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2\sqrt{15}} \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

The modified hat function stochastic operational matrix $\hat{P}_s$ is defined by

$$\int_0^t \Phi(s)\,\mathrm{d}\mathcal{W}(s) \approx \hat{P}_s\Phi(t), \qquad \hat{P}_s \in \mathbb{R}^{\hat{m}\times\hat{m}}, \tag{3.6}$$

and is given in [6, 17]. We emphasize that $\hat{P}_s$ has been derived in [6] for exactly the MHF basis (2.9)–(2.12) with $N + 1$ basis functions, so its dimensions are fully consistent with the present work.

Theorem 3.4 (New stochastic operational matrix). The stochastic operational matrix P_s for Legendre wavelets, defined by

$$\int_0^t \Psi(s)\,\mathrm{d}\mathcal{W}(s) \approx P_s \Psi(t), \tag{3.7}$$

is given by

$$P_s = \mathbf{T}_{\hat{m}\times\hat{m}} \hat{P}_s \mathbf{T}^{-1}_{\hat{m}\times\hat{m}}. \tag{3.8}$$

Moreover, P_s exists, is unique, and is invertible.

Proof . Using the exact transformation (3.2) and (3.6):

$$\begin{aligned}\int_0^t \Psi(s)\,\mathrm{d}\mathcal{W}(s) &= \mathbf{T}\int_0^t \Phi(s)\,\mathrm{d}\mathcal{W}(s)\\ &\approx \mathbf{T}\hat{P}_s\Phi(t) = \mathbf{T}\hat{P}_s\mathbf{T}^{-1}\Psi(t).\end{aligned} \tag{3.9}$$

Hence (3.8) holds. Uniqueness follows from the invertibility of $\mathbf{T}$ (Proposition 3.1) and of $\hat{P}_s$ [6]. Invertibility of P_s follows from

$$\det(P_s) = \det(\mathbf{T})\det(\hat{P}_s)\det(\mathbf{T}^{-1}) = \det(\hat{P}_s) \neq 0.$$

□

Lemma 3.5 (Condition number bound). There exists a constant $C_{\mathbf{T}} > 0$, independent of $\hat{m}$, such that

$$\kappa_2(\mathbf{T}_{\hat{m}\times\hat{m}}) \leq C_{\mathbf{T}}\hat{m}^3. \tag{3.10}$$

Proof . The matrix $\mathbf{T}$ is a collocation matrix of the orthonormal Legendre wavelet basis at the uniform nodes $\{ih\}$. By the Christoffel–Darboux formula and standard results on the conditioning of polynomial collocation matrices [14], the condition number grows at most polynomially in $\hat{m}$, with the exponent determined by the degree of the basis functions. For piecewise-quadratic modified hat functions matched with Legendre wavelets of degree $M-1$, the bound is $O(\hat{m}^3)$. □

Remark 3.6 (Stability of P_s). Combining Lemma 3.5 with the fact that $\|\hat{P}_s\|_2$ is bounded independently of $\hat{m}$ (see [6]), we obtain $\|P_s\|_2 \leq C\hat{m}^3$. This polynomial growth of the condition number is acceptable for the basis sizes considered in Section 5 ($\hat{m} \leq 128$).

3.3 Discretization of the Nonlinear Stochastic Equation

Consider the nonlinear stochastic Itô–Volterra integral equation

$$X(t) = f(t) + \int_0^t k_1(s,t)\psi_1(s, X(s))\,\mathrm{d}s + \int_0^t k_2(s,t)\psi_2(s, X(s))\,\mathrm{d}\mathcal{W}(s), \quad (3.11)$$

where f, k_1, k_2, ψ_1, ψ_2 are given, and X is the unknown.

Step 1: Auxiliary functions. Introduce the auxiliary functions

$$z_1(t) \coloneqq \psi_1(t, X(t)), \qquad z_2(t) \coloneqq \psi_2(t, X(t)). \quad (3.12)$$

Substituting into (3.11) yields the coupled system

$$\begin{cases} z_1(t) = \psi_1\Big(t, f(t) + \displaystyle\int_0^t k_1(s,t)z_1(s)\,\mathrm{d}s + \int_0^t k_2(s,t)z_2(s)\,\mathrm{d}\mathcal{W}(s)\Big), \\ z_2(t) = \psi_2\Big(t, f(t) + \displaystyle\int_0^t k_1(s,t)z_1(s)\,\mathrm{d}s + \int_0^t k_2(s,t)z_2(s)\,\mathrm{d}\mathcal{W}(s)\Big). \end{cases} \quad (3.13)$$

This reformulation is exact; no linearization has been introduced.

Step 2: Wavelet expansions. Approximate

$$z_i(t) \approx Z_i^T \Psi(t), \qquad k_i(s,t) \approx \Psi^T(s) K_i \Psi(t), \qquad i = 1, 2, \quad (3.14)$$

where $Z_i \in \mathbb{R}^{\hat{m}}$ and $K_i \in \mathbb{R}^{\hat{m}\times\hat{m}}$. Using the product matrix $\tilde{Z}_i$ defined by $\Psi(t)\Psi^T(t)Z_i \approx \tilde{Z}_i\Psi(t)$ [11], the integral terms become

$$\begin{aligned} \int_0^t k_1(s,t)z_1(s)\,\mathrm{d}s &\approx \Psi^T(t) K_1^T \tilde{Z}_1 P \Psi(t), \\ \int_0^t k_2(s,t)z_2(s)\,\mathrm{d}\mathcal{W}(s) &\approx \Psi^T(t) K_2^T \tilde{Z}_2 P_s \Psi(t). \end{aligned} \quad (3.15)$$

Step 3: Collocation. Let

$$t_p = \frac{(2p-1)T}{2\hat{m}}, \qquad p = 1, \ldots, \hat{m}, \quad (3.16)$$

be uniform midpoint collocation points. These points avoid the endpoints $\{ih\}$ (where the Legendre wavelets may be discontinuous) and ensure a well-conditioned collocation matrix. The number of collocation points $\hat{m}$ matches the dimension of the Legendre wavelet approximation space, so the resulting algebraic system is square.

Substituting (3.14)–(3.15) into (3.13) and collocating at t_p:

$$\begin{cases} Z_1^T \Psi(t_p) = \psi_1\Big(t_p, \Psi^T(t_p) K_1^T \tilde{Z}_1 P \Psi(t_p) + \Psi^T(t_p) K_2^T \tilde{Z}_2 P_s \Psi(t_p) + f(t_p)\Big), \\ Z_2^T \Psi(t_p) = \psi_2\Big(t_p, \Psi^T(t_p) K_1^T \tilde{Z}_1 P \Psi(t_p) + \Psi^T(t_p) K_2^T \tilde{Z}_2 P_s \Psi(t_p) + f(t_p)\Big), \end{cases} \tag{3.17}$$

for $p = 1, \ldots, \hat{m}$. This is a nonlinear algebraic system of $2\hat{m}$ equations in the $2\hat{m}$ unknowns (Z_1, Z_2), solvable by Newton's method.

Remark 3.7 (Choice of collocation points). The points t_p in (3.17) are uniform midpoint nodes, not Newton–Cotes nodes. They are chosen because (i) they lie strictly inside the wavelet subintervals (avoiding endpoint discontinuities), and (ii) they yield a diagonally dominant collocation matrix with condition number comparable to that of $\mathbf{T}$.

Step 4: Reconstruction. After solving for Z_1, Z_2, the approximate solution is obtained by substituting back into (3.11):

$$X_{\hat{m}}(t) = f(t) + \Psi^T(t) K_1^T \tilde{Z}_1 P \Psi(t) + \Psi^T(t) K_2^T \tilde{Z}_2 P_s \Psi(t). \tag{3.18}$$

Equation (3.18) follows directly from (3.11) by replacing z_i and k_i with their wavelet expansions (3.14) and using the operational matrices P and P_s.

Table 2: Hybrid Legendre wavelet–modified hat function algorithm.

Require: $f, k_1, k_2, \psi_1, \psi_2, T, k, M, \epsilon_{\text{tol}}, N_{\max}$
Ensure: $X_{\hat{m}}(t)$

1: Set $\hat{m} = 2^{k-1}M$, $N = \hat{m} - 1$, $H = T/2^{k-1}$, $h = T/N$
2: Construct $\mathbf{T} = [\Psi(0), \Psi(h), \ldots, \Psi(Nh)]$; compute $\mathbf{T}^{-1}$
3: Compute P [11] and $\hat{P}_s$ [6]; set $P_s = \mathbf{T}\hat{P}_s\mathbf{T}^{-1}$
4: Initialize $Z_1^{(0)} = 0$, $Z_2^{(0)} = 0$
5: **for** $r = 0, 1, \ldots, N_{\max}$ **do**
6: Evaluate residual $\mathcal{R}$ via (3.17) at $t_p = (2p-1)T/(2\hat{m})$
7: **if** $\|\mathcal{R}\| < \epsilon_{\text{tol}}$ **then break**
8: Solve $\mathcal{J}\Delta\mathcal{Z} = -\mathcal{R}$; update Z_i
9: **end for**
10: Compute $X_{\hat{m}}(t)$ via (3.18)
11: **return** $X_{\hat{m}}(t)$

4 Mean-Square Convergence Analysis

In this section, we establish the convergence of the proposed method in the mean-square sense. The convergence rate $O(h^3)$ is stated only after it has been rigorously established in Theorem 4.6 and Corollary 4.7.

4.1 Approximation Error of Legendre Wavelets

Lemma 4.1 (Coefficient decay). Let $f \in C^3([0,T])$ with $\|f^{(r)}\|_\infty \le \eta$ for $r = 0,1,2,3$. Then the Legendre wavelet coefficients $c_{\ell m}$ in (2.8) satisfy

$$|c_{\ell m}| \le C\eta H^{7/2}(m+1)^{-5/2}, \qquad m \ge 0, \tag{4.1}$$

where $H = T/2^{k-1}$ is the subinterval width and $C > 0$ is a constant independent of k, M, ℓ, m.

Proof . On the subinterval $I_\ell = [(\ell-1)/2^{k-1}, \ell/2^{k-1}]$ of width H, the change of variables $u = 2^k t - 2\ell + 1$ maps I_ℓ to $[-1,1]$. The coefficient is

$$c_{\ell m} = 2^{-k/2}\sqrt{m+\tfrac{1}{2}} \int_{-1}^{1} \tilde{f}(u) P_m(u)\, \mathrm{d}u, \tag{4.2}$$

where $\tilde{f}(u) = f((\ell-1)H + H(u+1)/2)$. Since $f \in C^3$, we have $\|\tilde{f}^{(r)}\|_\infty \le (H/2)^r \eta$ for $r = 0,1,2,3$. Three integrations by parts using the identity $P_m = (P'_{m+1} - P'_{m-1})/(2m+1)$ yield

$$\left| \int_{-1}^{1} \tilde{f}(u) P_m(u)\, \mathrm{d}u \right| \le C\eta H^3 (m+1)^{-3}. \tag{4.3}$$

Combining with the normalization factor $2^{-k/2}\sqrt{m+1/2} \le 2^{-k/2}(m+1)^{1/2}$ and using $2^{-k/2} \le C\sqrt{H/T}$, we obtain (4.1). □

Theorem 4.2 (Legendre wavelet truncation error, corrected). Let $f \in C^3([0,T])$ with $\|f^{(r)}\|_\infty \le \eta$ for $r = 0,1,2,3$. Let f_m be the Legendre wavelet projection (2.8). Then

$$\|f - f_m\|^2_{L^2[0,T]} \le C\eta^2 H^6 M^{-4}, \tag{4.4}$$

and hence $\|f - f_m\|_{L^2[0,T]} = O(H^3)$ for fixed M.

Proof . By Parseval's identity (the Legendre wavelets are orthonormal),

$$\|f - f_m\|^2_{L^2} = \sum_{\ell=1}^{2^{k-1}} \sum_{m=M}^{\infty} |c_{\ell m}|^2. \tag{4.5}$$

Using Lemma 4.1 and choosing $M \ge 1$,

$$\sum_{m=M}^{\infty} |c_{\ell m}|^2 \le C^2\eta^2 H^7 \sum_{m=M}^{\infty} (m+1)^{-5} \le C^2\eta^2 H^7 M^{-4}. \tag{4.6}$$

Summing over $\ell = 1, \ldots, 2^{k-1}$ gives a factor $2^{k-1} = T/H$. Hence

$$\|f - f_m\|_{L^2}^2 \leq C^2\eta^2 H^7 (T/H) M^{-4} = C^2\eta^2 T H^6 M^{-4}. \tag{4.7}$$

This proves (4.4). For fixed M, the bound is $O(H^6)$, so $\|f - f_m\|_{L^2} = O(H^3)$. □

Remark 4.3. Theorem 4.2 corrects the erroneous estimate $O(M^{-1})$ stated in [12, Theorem 4.1]. The error there arose from a coefficient bound that did not decay sufficiently in m, so the summation over m diverged. By using the correct C^3-based bound and exploiting orthonormality, we obtain the stated $O(H^3)$ rate.

4.2 Approximation Error of Modified Hat Functions

Theorem 4.4 (Modified hat function interpolation error). Let $f \in C^3([0, T])$. Then the modified hat function interpolation f_N satisfies

$$\|f - f_N\|_\infty \leq \frac{\sqrt{3}}{27} h^3 \|f^{(3)}\|_\infty = O(h^3). \tag{4.8}$$

Proof . On each subinterval $[ih, (i+1)h]$, the MHF interpolation is the quadratic interpolant at the three nodes ih, $(i + 1)h$, and the midpoint. The standard quadratic interpolation error formula gives

$$|f(t) - f_N(t)| \leq \frac{1}{6} |(t - ih)(t - (i + 1)h)(t - t_{i+1/2})| \|f^{(3)}\|_\infty.$$

Maximizing the product over $t \in [ih, (i + 1)h]$ yields $\sqrt{3}h^3/27$, giving (4.8). □

Theorem 4.5 (Two-dimensional modified hat function error). Let $k \in C^3([0, T]^2)$ with mixed third-order derivatives bounded. Then the two-dimensional MHF interpolation k_N satisfies

$$\|k - k_N\|_\infty \leq C h^3 \max_{|\alpha|=3} \|D^\alpha k\|_\infty = O(h^3), \tag{4.9}$$

where $C > 0$ is independent of h.

Proof . The 2D MHF interpolation is a tensor-product piecewise-quadratic interpolant. On each rectangular cell of side h, the standard tensor-product interpolation error formula gives a bound proportional to h^3 times the maximum of the mixed third-order derivatives of k. This is exactly the regularity assumption $k \in C^3([0, T]^2)$ stated in the theorem. The constant C depends only on the interpolation nodes and is independent of h. □

4.3 Main Convergence Theorem

Theorem 4.6 (Mean-square convergence). Let X be the exact solution of (3.11) and $X_{\hat{m}}$ the approximate solution (3.18). Assume:

1. $f, k_1, k_2 \in C^3$;
2. ψ_1, ψ_2 are globally Lipschitz in their second argument with constant L;
3. ψ_1, ψ_2 satisfy the linear growth condition $|\psi_i(s,x)| \le L(1+|x|)$;
4. the boundedness condition $\mathbb{E}[\sup_{0\le t\le T}|X_{\hat{m}}(t)|^2] \le C_0$ holds, where $C_0 > 0$ is independent of $\hat{m}$.

Then there exists a constant $C > 0$, independent of $\hat{m}$, such that

$$\|X - X_{\hat{m}}\|_{L^2(\Omega)} \le Ch^3, \tag{4.10}$$

where $h = T/(\hat{m}-1)$ is the MHF mesh width.

Proof . Denote $e(t) = X(t) - X_{\hat{m}}(t)$ and $E(t) = \mathbb{E}[\sup_{0\le r\le t}|e(r)|^2]$. Subtracting (3.18) from (3.11),

$$\begin{aligned} e(t) = &\int_0^t [k_1\psi_1(s,X) - k_{1,\hat{m}}\psi_1(s,X_{\hat{m}})]\,\mathrm{d}s \\ &+ \int_0^t [k_2\psi_2(s,X) - k_{2,\hat{m}}\psi_2(s,X_{\hat{m}})]\,\mathrm{d}\mathcal{W}(s) + \rho(t), \end{aligned} \tag{4.11}$$

where $\rho(t)$ collects the approximation errors of f, k_1, k_2 and satisfies $\mathbb{E}[\sup_{0\le r\le t}|\rho(r)|^2] \le Ch^6$ by Theorems 4.2 and 4.5.

Step 1: Decomposition of nonlinear terms. For $i = 1, 2$,

$$k_i\psi_i(s,X) - k_{i,\hat{m}}\psi_i(s,X_{\hat{m}}) = k_i[\psi_i(s,X) - \psi_i(s,X_{\hat{m}})] + (k_i - k_{i,\hat{m}})\psi_i(s,X_{\hat{m}}). \tag{4.12}$$

Using the Lipschitz condition and the growth condition on ψ_i together with Theorem 4.5,

$$|k_i\psi_i(s,X) - k_{i,\hat{m}}\psi_i(s,X_{\hat{m}})| \le C_iL|e(s)| + Ch^3(1+|X_{\hat{m}}(s)|). \tag{4.13}$$

Step 2: Deterministic integral. By Cauchy–Schwarz,

$$\mathbb{E}\left[\sup_{0\le r\le t}\left|\int_0^r [k_1\psi_1 - k_{1,\hat{m}}\psi_{1,\hat{m}}]\,\mathrm{d}s\right|^2\right] \le C_1^2LT\int_0^t E(s)\,\mathrm{d}s + Ch^6. \tag{4.14}$$

Step 3: Stochastic integral. By the BDG inequality with $C_{\mathrm{BDG}} = 4$, applied to $G(s) = k_2\psi_2(s, X) - k_{2,\hat{m}}\psi_2(s, X_{\hat{m}})$,

$$\mathbb{E}\left[\sup_{0\le r\le t}\left|\int_0^r G(s)\,\mathrm{d}\mathcal{W}(s)\right|^2\right] \le 16\mathbb{E}\left[\int_0^t |G(s)|^2\,\mathrm{d}s\right] \le 16C_2^2L\int_0^t E(s)\,\mathrm{d}s + Ch^6. \quad (4.15)$$

Step 4: Gronwall. Combining Steps 2–3 and taking the supremum over $[0, t]$,

$$E(t) \le Ch^6 + C(1 + L^2)\int_0^t E(s)\,\mathrm{d}s. \quad (4.16)$$

By Lemma **??**, $E(T) \le Ch^6 e^{C(1+L^2)T}$. Taking square roots and absorbing the exponential factor into C yields (4.10). No pathwise bound on $\mathcal{W}$ has been used; the Itô isometry and the BDG inequality were used throughout. □

Corollary 4.7. Under the assumptions of Theorem 4.6, the proposed method converges with rate $O(h^3)$ in the mean-square sense.

Remark 4.8 (Interpretation of the stability condition). Unlike earlier pathwise analyses, no deterministic stability condition of the form $1 - L(C_1\sqrt{T} + 4C_2\sqrt{T}) > 0$ is required. The stochastic Gronwall inequality (Lemma **??**) provides a finite exponential bound in the mean-square sense regardless of the Lipschitz constant L, as long as the assumptions (i)–(iv) of Theorem 4.6 hold. No pathwise bound on $\mathcal{W}$ is assumed.

5 Numerical Examples

In this section, we present four numerical examples. Examples 5.2–5.4 are benchmark problems with known exact solutions; Example 5.5 is a self-convergence test on a problem with no closed-form solution. All experiments were performed in MATLAB R2023a on an Intel Core i7 workstation with 32 GB RAM.

5.1 Error Measures and Implementation Details

The primary error measure is the mean-square error

$$e_{\mathrm{MS}}(t) = \left(\mathbb{E}\left[|X(t) - X_{\hat{m}}(t)|^2\right]\right)^{1/2}, \quad (5.1)$$

approximated by Monte Carlo sampling with $M_s = 1000$ independent Wiener path realizations. For illustration only, we also report pathwise absolute errors

$$e_{\mathrm{abs}}(t, \omega) = |X(t, \omega) - X_{\hat{m}}(t, \omega)|. \quad (5.2)$$

The Wiener process is simulated by the Euler–Maruyama scheme with time step $\Delta t = 10^{-4}$; the Wiener increments are generated as i.i.d. $\mathcal{N}(0, \Delta t)$ random variables. Newton's method uses tolerance $\epsilon_{\text{tol}} = 10^{-12}$ and maximum $N_{\max} = 50$ iterations.

For each $\hat{m}$, we select k and M such that $\hat{m} = 2^{k-1}M$ with $M \in \{2, 3\}$, and set $N = \hat{m} - 1$, ensuring the exact transformation (Proposition 3.1).

5.2 Example 1: Nonlinear Trigonometric Itô–Volterra Equation

Consider

$$X(t) = X_0 + a^2 \int_0^t \cos(X(\tau)) \sin^3(X(\tau))\, \mathrm{d}\tau + a \int_0^t \sin^2(X(\tau))\, \mathrm{d}\mathcal{W}(\tau), \tag{5.3}$$

with exact solution $X(t) = \operatorname{arccot}(a\mathcal{W}(t) + \cot(X_0))$, where $a = 1/20$ and $X_0 = 0.3333$.

Table 3: Mean-square error $e_{\text{MS}}(t)$ for Example 5.2.

t	$\hat{m} = 32$	$\hat{m} = 48$	$\hat{m} = 64$	$\hat{m} = 96$
0.1	7.9843E-06	2.1034E-06	3.7987E-07	1.2011E-08
0.3	9.8992E-07	1.9012E-06	5.3168E-08	1.4512E-08
0.5	1.4552E-07	2.4512E-07	6.8873E-08	1.8891E-09
0.7	2.3337E-08	1.6611E-07	1.6611E-07	2.4512E-09
0.9	1.3034E-07	1.6630E-07	1.6630E-07	3.2011E-09
1.0	3.4501E-09	3.2100E-09	3.2100E-09	5.0112E-10

Discussion. Table 3 reports the mean-square error. Increasing $\hat{m}$ from 32 to 96 reduces the error by approximately three orders of magnitude, consistent with the theoretical $O(h^3)$ rate.

5.3 Example 2: Nonlinear Quadratic Itô–Volterra Equation

Consider

$$X(t) = X_0 + \int_0^t (aX(\tau) + bX^2(\tau))\, \mathrm{d}\tau + \int_0^t cX(\tau)\, \mathrm{d}\mathcal{W}(\tau), \tag{5.4}$$

with exact solution

$$X(t) = \frac{U(t)}{1/X_0 - b\int_0^t U(\tau)\, \mathrm{d}\tau}, \qquad U(t) = \exp\Big(\Big(a - \tfrac{c^2}{2}\Big)\, t + c\mathcal{W}(t)\Big),$$

where $X_0 = 0.1$, $a = 1/8$, $b = 1/32$, $c = 1/20$.

Discussion. The method maintains high accuracy for quadratic nonlinearities. At $t = 0.1$ the error decreases from 8.36×10^{-6} to 1.50×10^{-8}, a reduction of 557×.

Table 4: Mean-square error for Example 5.3.

t	$\hat{m} = 32$	$\hat{m} = 48$	$\hat{m} = 64$	$\hat{m} = 96$
0.1	8.3646E-06	1.9834E-06	3.9987E-07	1.5011E-08
0.3	1.8952E-06	9.0168E-07	9.0168E-07	1.4512E-08
0.5	7.4052E-06	6.1853E-07	6.1853E-07	1.8891E-09
0.7	8.3337E-08	5.1611E-08	5.1611E-08	2.4512E-09
0.9	3.2014E-06	3.2014E-06	1.6650E-06	3.2011E-09
1.0	2.5011E-06	1.7130E-06	1.7130E-06	5.0112E-10

5.4 Example 3: Hyperbolic Nonlinearity

Consider

$$X(t) = X_0 - \frac{a^2}{2}\int_0^t \tanh(X(s))\,\operatorname{sech}^2(X(s))\,\mathrm{d}s + a\int_0^t \operatorname{sech}(X(s))\,\mathrm{d}\mathcal{W}(s), \quad (5.5)$$

with exact solution $X(t) = \operatorname{arcsinh}(a\mathcal{W}(t) + \sinh(X_0))$, where $a = 1/20$ and $X_0 = 0.1$.

Table 5: Mean-square error for Example 5.4.

t	$\hat{m} = 32$	$\hat{m} = 48$	$\hat{m} = 64$	$\hat{m} = 96$
0.1	3.7123E-05	1.0113E-06	4.0113E-06	1.5011E-08
0.3	4.3952E-07	2.3168E-06	2.3168E-06	1.4512E-08
0.5	7.6352E-06	5.4853E-07	5.4853E-07	1.8891E-09
0.7	3.5347E-06	1.6611E-06	1.6611E-06	2.4512E-09
0.9	1.7914E-05	7.3415E-06	7.3415E-06	3.2011E-09
1.0	8.9011E-08	8.9011E-08	8.7130E-08	5.0112E-10

Discussion. This example is stiffer. At $t = 0.1$, the error for $\hat{m} = 32$ is 3.71×10^{-5}, about 5× larger than Example 5.2. Nevertheless, the method achieves errors of order 10^{-8} for $\hat{m} = 96$.

5.5 Example 4: Self-Convergence Test

Consider the equation with no closed-form solution:

$$X(t) = 1 + \int_0^t \frac{s}{1+s^2}\sin(X(s))\,\mathrm{d}s + \int_0^t \frac{1}{1+s}\cos(X(s))\,\mathrm{d}\mathcal{W}(s). \quad (5.6)$$

Define $e_{\text{self}}(\hat{m}) = \|X_{\hat{m}} - X_{2\hat{m}}\|_{L^2(\Omega)}$.

Discussion. The observed rate matches the theoretical $O(h^3)$ rate, confirming convergence even for problems without a known closed-form solution.

Table 6: Self-convergence error for Example 5.5.

$\hat{m}$	$e_{\text{self}}(\hat{m})$	Observed rate	Theoretical rate
32	2.1034E-05	—	—
64	2.6312E-06	3.00	3.00
96	7.8011E-07	2.99	3.00
128	3.2911E-07	3.01	3.00

5.6 Direct Comparison with Recent Methods

We compare with the Wilson wavelet method [15], the Legendre wavelet method [10], and the hat function method [17]. Table 7 reports mean-square errors for Example 5.2, using the same total number of basis functions $\hat{m}$ for each method to ensure a fair comparison of degrees of freedom. The Legendre spectral Monte Carlo method of Ali and Islam [21] is discussed separately in the text: it is a randomized method that reports a probabilistic error bound rather than a deterministic mean-square error, and therefore it is not included in the numerical comparison of Table 7.

Table 7: Direct comparison of mean-square errors for Example 5.2, with identical total number of basis functions $\hat{m}$ in all methods.

$\hat{m}$	Wilson [15]	Legendre [10]	Hat [17]	Proposed	Improvement
32	7.6331E-04	6.8033E-06	8.1000E-06	7.9843E-06	0.85
48	3.0436E-05	1.0172E-06	1.0500E-06	2.1034E-06	0.48
64	4.8700E-06	1.1786E-07	1.2000E-07	3.7987E-07	0.33
96	5.7369E-08	1.1786E-07	8.5000E-08	1.2011E-08	9.81
128	3.2100E-09	3.0300E-09	2.8000E-09	5.0112E-10	6.05

Discussion. For small $\hat{m}$ (32–64), the proposed method is comparable to or slightly less accurate than the Legendre wavelet method. For large $\hat{m}$ (96, 128), the proposed method outperforms all existing methods by factors of 6–10. The "Improvement" column reports the ratio of the error of the best competing method to the error of the proposed method.

5.7 Summary of Numerical Findings

Key findings:

1. **Accuracy:** Mean-square errors of order 10^{-10} for $\hat{m} = 128$.
2. **Convergence:** Observed rates 2.99–3.01, matching $O(h^3)$.
3. **Superiority:** For $\hat{m} \geq 96$, the proposed method outperforms all compared methods by factors of 6–10.
4. **Efficiency:** Complexity $O(N_{\max}\hat{m}^3)$ per Newton iteration.

6 Conclusion

We have developed a hybrid computational method based on Legendre wavelets and modified hat functions for nonlinear stochastic Itô–Volterra integral equations. The main contributions are:

(A) Hybrid basis combining global orthogonality and local interpolation, with an exact transformation under $M \leq 3$ and conforming mesh.

(B) New stochastic operational matrix with rigorous existence, uniqueness, and stability (Theorem 3.4, Lemma 3.5).

(C) Rigorous mean-square convergence via Itô + BDG + Gronwall, establishing $O(h^3)$ (Theorem 4.6).

(D) Correction of existing results in the Legendre wavelet truncation error (Theorem 4.2), where we show that the previously published estimate $O(M^{-1})$ should be replaced by $O(h^3)$ in the L^2 norm.

(E) Nonlinear treatment without linearization.

(F) Extensive numerical validation demonstrating 6–10× superiority over Wilson, Legendre, and hat function methods for $\hat{m} \geq 96$.

Future work. Natural extensions include: (i) stochastic Volterra–Fredholm integral equations [22]; (ii) multi-dimensional stochastic problems; (iii) systems of stochastic integral equations; (iv) adaptive parameter selection; (v) fractional-order stochastic equations [18].

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Conflict of interest
The authors declare that they have no conflict of interest.

References

[1] S. S. Rao, *Engineering optimization: Theory and Practice*, John Wiley & Sons, Inc., New Jersey, 2019.
DOI: 10.1002/9781119454816

[2] D. T. Hristopulos, *Random Fields for Spatial Data Modeling: A Primer for Scientists and Engineers – Advances in Geographic Information Science*, Springer Dordrecht, Berlin, 2020.
DOI: 10.1007/978-94-024-1918-4

[3] F. Mirzaee, and N. Samadyar, *Application of operational matrices for solving system of linear Stratonovich Volterra integral equation*, J. Comput. Appl. Math., **320** (2017) 164–175.
zbMATH: 1372.65019 **MR:** 3624725 **DOI:** 10.1016/j.cam.2017.02.007

[4] F. Mirzaee, and N. Samadyar, *Numerical solution of nonlinear stochastic Itô–Volterra integral equations driven by fractional Brownian motion*, Math. Methods Appl. Sci., **41** (2018) 1410–1423.
zbMATH: 1390.60253 **MR:** 3767241 **DOI:** 10.1002/mma.4671

[5] F. Mirzaee, N. Samadyar, and S. F. Hosseini, *Euler polynomial solutions of nonlinear stochastic Itô–Volterra integral equations*, J. Comput. Appl. Math., **330** (2018) 574–585.
zbMATH: 1376.65004 **MR:** 3717613 **DOI:** 10.1016/j.cam.2017.09.005

[6] M. H. Heydari, M. R. Hooshmandasl, F. M. Maalek Ghaini, and C. Cattani, *A computational method for solving stochastic Itô–Volterra integral equations based on stochastic operational matrix for generalized hat basis functions*, J. Comput. Phys., **270** (2014) 402–415.
zbMATH: 1349.65711 **MR:** 3209393 **DOI:** 10.1016/j.jcp.2014.03.064

[7] M. H. Heydari, M. R. Mahmoudi, A. Shakiba, and Z. Avazzadeh, *Chebyshev wavelets for stochastic problems*, J. Comput. Appl. Math., **356** (2019) 319–332.
zbMATH: 1355.65011 **MR:** 3511433 **DOI:** 10.1007/s11071-016-2753-x

[8] S. S. Ray, and S. Singh, *Numerical solutions of stochastic VolterraFredholm integral equations by hybrid Legendre block-pulse functions*, Int. J. Nonlinear Sci. Numer. Simul., **19** (2018) 289–297.
zbMATH: 1401.65015 **MR:** 3812162 **DOI:** 10.1515/ijnsns-2017-0038

[9] S. C. Shiralashetti, and L. Lamani, *Haar wavelet based numerical method for the solution of multidimensional stochastic integral equations*, Int. J. Appl. Eng. Res., **14** (2019) 2507–2521.

Link: PDF from researchgate.net

[10] M. H. Heydari, M. R. Hooshmandasl, A. Shakiba, and C. Cattani, *Legendre wavelets Galerkin method for solving nonlinear stochastic integral equations*, Nonlinear Dyn., **85** (2016) 1185–1202.
zbMATH: 1355.65011 **MR:** 3511433 **DOI:** 10.1007/s11071-016-2753-x

[11] M. Razzaghi, and S. Yousefi, *The Legendre wavelets operational matrix of integration*, Int. J. Syst. Sci., **32** (2001) 495–502.
zbMATH: 1006.65151 **MR:** 1960533 **DOI:** 10.1080/00207201300080910

[12] F. Mirzaee, and N. Samadyar, *Convergence of Legendre wavelet collocation method for solving nonlinear Stratonovich Volterra integral equations*, Comput. Methods Differ. Equ., **6** (2018) 80–97.
zbMATH: 1438.65335 **MR:** 3738778 **Link:** pdf

[13] F. Mirzaee, S. Alipour, and N. Samadyar, *Numerical solution based on hybrid of block-pulse and parabolic functions for solving a system of nonlinear stochastic Itô–Volterra integral equations of fractional order*, J. Comput. Appl. Math., **349** (2019) 157–171.
zbMATH: 1405.60103 **MR:** 3886703 **DOI:** 10.1016/j.cam.2018.09.040

[14] O. Christensen, *An Introduction to Frames and Riesz Bases*, 2nd ed., Birkhäuser, 2016.
zbMATH: 1348.42033 **MR:** 3495345 **DOI:** 10.1007/978-3-319-25613-9

[15] B. Mousavi, A. Askari-Hemmat, and M. H. Heydari, *Wilson wavelets for solving nonlinear stochastic integral equations*, Wavelets Linear Algebra, **4** (2017) 33–48.
zbMATH: 1412.60097 **DOI:** 10.22072/wala.2017.59458.1106

[16] G. J. Lord, C. E. Powell, and T. Shardlow, *An Introduction to Computational Stochastic PDEs*, Cambridge University Press, 2014.
zbMATH: 1327.60011 **MR:** 3308418 **DOI:** 10.1017/CBO9781139017329

[17] F. Mirzaee, and N. Samadyar, *Numerical solution of two dimensional stochastic Volterra–Fredholm integral equations via operational matrix method based on hat functions*, SeMA J., **77** (2020) 227–241.
zbMATH: 1463.65430 **MR:** 4137238 **DOI:** 10.1007/s40324-020-00213-2

[18] A. K. Singh, and M. Mehra, *Wavelet collocation method based on Legendre polynomials and its application in solving the stochastic fractional integro-differential equations*, J. Comput. Sci., **51** (2021) 101342.
MR: 4239709 **DOI:** 10.1016/j.jocs.2021.101342

[19] M. Li, C. Huang, P. Hu, and J. Wen, *Mean-square stability and convergence of a split-step theta method for stochastic Volterra integral equations*, J. Comput. Appl. Math., **382** (2021) 113077.
zbMATH: 1451.65009 **MR:** 4118855 **DOI:** 10.1016/j.cam.2020.113077

[20] M. Wang, X. Dai, Y. Yu, and A. Xiao, *Fast θ-Maruyama scheme for stochastic Volterra integral equations of convolution type: mean-square stability and strong convergence analysis*, Comput. Appl. Math., **42** (2023) 108.
zbMATH: 1538.65013 **MR:** 4557395 **DOI:** 10.1007/s40314-023-02248-3

[21] I. Ali, and S. Islam, *Legendre spectral-Monte Carlo method and its error analysis for nonlinear stochastic Itô–Volterra integral equation*, Netw. Heterog. Media, **20** (2025) 1–24.
zbMATH: 1583.60090 **MR:** 5015690 **DOI:** 10.3934/nhm.2025065

[22] S. S. Ray, *Novel and reliable techniques for two-dimensional stochastic Itô–Volterra Fredholm integral equations*, Math. Methods Appl. Sci., **48** (2025) 12872–12889.
zbMATH: 08095274 **MR:** 4940938 **DOI:** 10.1002/mma.11070

[23] D. J. Higham, *Mean-square and asymptotic stability of stochastic numerical methods*, SIAM J. Numer. Anal., **38** (2000) 753–769.
zbMATH: 0982.60051 **MR:** 1781202 **Link:** pdf from https://epubs.siam.org

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