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© Research paper

Ascending chains in Artinian lattices

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Abstract

It is well-known that every Artinian lattice is Loewy but not necessarily Noetherian. For a compactly generated Artinian lattice we determine its deviation from being Noetherian in terms of its Loewy length: we show that a compactly generated Artinian lattice L is λ-Noetherian, where λ is the least regular cardinal greater than the Loewy length of L. Along the way we develop the basic theory of λ-Noetherian lattices (for λ a regular cardinal) in the setting of compactly generated lattices, relate the invariants Gdim, thick and g of a lattice to its Loewy length, and, as an application of these techniques, obtain a lattice-theoretic generalization of a theorem of Armendariz on rings with the descending chain condition on essential (left) ideals.

1 Introduction

It is well-known that every Artinian module over commutative rings is countably generated, see [4]. Karamzadeh [4] has characterized rings over which Artinian modules

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are countably generated. Karamzadeh and Motamedi [5] have found a nontrivial upper bound for the cardinality of a generating set of an Artinian module over any ring. Shirali [10] has extended the latter upper bound to modules with Krull dimension. We should also remind the reader that Artinian modules over commutative rings have a finite Noetherian dimension (i.e., the dual of Krull dimension), and recently, it was shown in [6] that the countability of the Noetherian dimension of any module M over any ring forces M to be countably generated. Our main aim in this article is to extend some of these results (for example, the main result of [5]) to lattices. The theory of modular lattices is one of the tools dealing with the abstract aspect of the inclusion between submodules; see, for example, [1], [2], [9]. Following this line of thought, we investigate the lattice concept of λ-Noetherianness, which was originally introduced by R.S. Pierce in [8] and by A. Simis in [11] for modules and extensively studied by O.A.S. Karamzadeh and M. Motamedi in [5]. We obtain the basic properties of λ-Noetherian lattices, where λ is a regular cardinal number ($\aleph_0$-Noetherian lattices are just the Noetherian ones). Our main result is that an Artinian lattice L is λ-Noetherian, where λ is the least regular cardinal greater than the Loewy length of L.

2 Preliminaries

As we are mainly interested in the subject with references to modules, the terms we use in this paper are taken from the theory of modules. For the basic notation and undefined terms, we refer the reader to [7].

Definition 2.1. A subset A of a lattice L is said to be directed if for all $a, b \in A$ there exists an element $c \in A$ such that $a \leq c$ and $b \leq c$.

Definition 2.2. A complete lattice L is said to be upper-continuous, provided that for each directed subset A of L and any $a \in L$, we have $a \wedge (\bigvee_{x\in A} x) = \bigvee_{x\in A}(a \wedge x)$.

Definition 2.3. An element a in an upper-continuous lattice L is said to be compact if for each directed subset A of L, for which $a \leq \bigvee_{x\in A} x$, there exists an element $x_0 \in A$ such that $a \leq x_0$. L is compact if 1 is compact in L, and L is compactly generated if each element of L is a join of compact elements.

Throughout this note, L will always denote an upper-continuous and modular lattice with 0 and 1 ($0 \neq 1$). For all $a \leq b$,

$$= \{x \in L : a \leq x \leq b\}$$

is called the factor of b by a. Obviously, $[a, b]$ is a sublattice of L and $[0, 1] = L$. The mapping $\phi_b : x \to x \wedge b$ is an isomorphism of $[a, a \vee b]$ onto $[a \wedge b, b]$; see [7], Proposition 1.3.1. The inverse isomorphism is $\psi_a : x \to x \vee a$. We shall also use the following cancellation property of modular lattices:

$$x \leq y, \quad x \wedge a = y \wedge a, \quad x \vee a = y \vee a \implies x = y. \tag{2.1}$$

Indeed, by modularity, $x = x \vee (x \wedge a) = x \vee (y \wedge a) = (x \vee a) \wedge y = (y \vee a) \wedge y = y$.

A subset I of $L\backslash\{0\}$ is called join-independent if for any finite subset X of I and $x \in I\backslash X$, we have $\bigvee X \wedge x = 0$, where $\bigvee X$ denotes the join of all elements of X.

Definition 2.4. The Goldie dimension of a lattice L, denoted by $\operatorname{Gdim} L$, is defined to be the supremum of all cardinal numbers $\aleph$ such that there exists a join-independent set X of cardinality $\aleph$.

Definition 2.5. A nonzero element $a \in L$ in a lattice L is essential in L if for any nonzero element $x \in L$, $a \wedge x \neq 0$.

For the proof of the following, see [7], Section 1.5.

Lemma 2.6.
1. If $\operatorname{Gdim} L = n < \infty$ and $a \in L$, then $\operatorname{Gdim}[0, a] \leq n$, and equality holds if and only if a is essential in L.
2. If $\operatorname{Gdim} L = n < \infty$ and the set $\{a_1, a_2, \ldots, a_k\}$ of elements of L is join-independent, then

$$\operatorname{Gdim}[0, a_1 \vee a_2 \vee \cdots \vee a_k] = \sum_{i=1}^{k} \operatorname{Gdim}[0, a_i].$$

An element $a \in L$ is called an atom if $b \leq a$ implies either $b = 0$ or $b = a$, or equivalently if $[0, a]$ is simple. The join of all atoms of L, which is denoted by $s(L)$, is called the socle of L, and whenever 1 is a join of atoms, we call L a semiatomic lattice. It is clear that $[0, s(L)]$ is semiatomic. Intervals of semiatomic (modular) lattices are semiatomic. Every atom of L is compact: if a is an atom, A is directed and $a \leq \bigvee A$, but $a \not\leq x$ for all $x \in A$, then $a \wedge x = 0$ for all $x \in A$, and upper-continuity gives $a = a \wedge \bigvee A = \bigvee_{x \in A}(a \wedge x) = 0$, a contradiction.

Definition 2.7. The Loewy chain of a lattice L is defined inductively: $s_0(L) = 0$, $s_1(L) = s(L)$; if $\alpha = \beta + 1$, put $s_\alpha(L) = s([s_\beta(L), 1])$, and if α is a limit ordinal, we define $s_\alpha(L) = \bigvee_{\beta<\alpha} s_\beta(L)$. So we obtain

$$0 = s_0(L) \leq s_1(L) \leq \cdots \leq s_\alpha(L) \leq \cdots .$$

Since L is a set, there exists a least ordinal α such that

$$s_\alpha(L) = s_{\alpha+1}(L) = \cdots .$$

This ordinal is called the Loewy length of L and is denoted by $l(L)$. If $s_{l(L)}(L) = 1$, L is called a Loewy lattice.

Note that $s_\gamma(L) < s_{\gamma+1}(L)$ for all $\gamma < l(L)$, so the Loewy chain is strictly ascending up to $l(L)$.

Definition 2.8. The Krull dimension of a partially ordered set A, denoted by K-dim A, is defined inductively as follows:

(a) If A is trivial, then K-dim $A = -\infty$.
(b) If A is Artinian, then K-dim $A = 0$.
(c) For a general ordinal α, we define K-dim $A = \alpha$ provided that:
  (i) there is no ordinal $\beta < \alpha$ such that K-dim $A = \beta$;
  (ii) in any descending chain of elements of A, all but finitely many factors have Krull dimension less than α.

For the proofs of the next results, see [7], Sections 3.1 and 3.2.

Lemma 2.9. If B is a subset of A, and A has Krull dimension, then B also has Krull dimension and K-dim $B \leq$ K-dim A.

Theorem 2.10. Let L be a lattice (modular and upper-continuous). If L has Krull dimension, then L has finite Goldie dimension.

3 λ-Noetherian lattices

Definition 3.1. Let L be a lattice and $a \neq 0$ be an element of it. A nonempty set S of compact elements of L is called a generating set of a if $a = \bigvee_{s\in S} s$. The least cardinal number in the set of cardinalities of all generating sets of a is denoted by $g(a)$ (we put $g(0) = 1$). By $g(L)$ we mean $g(1)$, see [4], [5] and [10]. For an interval $[x, y]$ of L, $g[x, y]$ denotes $g(y)$ computed in the lattice $[x, y]$ (with compactness defined relative to $[x, y]$).

As in the module case, from now on in this section L is assumed to be compactly generated (as is the lattice of submodules of any module); then every interval of L is again compactly generated, and every nonzero element has a generating set.

Definition 3.2. A lattice L is λ-Noetherian if λ is the least regular cardinal such that $g(a) < \lambda$ for all $a \in L$.

Here and below, regular cardinals are infinite regular cardinals; $\aleph_0$ is the least infinite cardinal and $\aleph_1$ the least uncountable one. A chain $\{a_\alpha : \alpha < \kappa\}$ is called *stable* if there is some $\beta < \kappa$ such that $a_\alpha = a_\beta$ for all $\alpha \geq \beta$.

The following is the lattice counterpart of [8], Lemma 1.1.

Lemma 3.3. Let L be a lattice and κ a regular cardinal. The following are equivalent:

(i) $g(a) < \kappa$ for all $a \in L$;
(ii) every well-ordered ascending chain $\{a_\alpha : \alpha < \kappa\}$ of elements of L is stable.

In particular, L is λ-Noetherian if and only if λ is the least regular cardinal such that every well-ordered ascending chain $\{a_\alpha : \alpha < \lambda\}$ of elements of L is stable. Moreover, if (ii) holds for κ, it holds for every regular cardinal $\kappa' \geq \kappa$.

Proof . (i)⇒(ii). Let $\{a_\alpha : \alpha < \kappa\}$ be an ascending chain and $a = \bigvee_{\alpha<\kappa} a_\alpha$; we may assume $a \neq 0$. Since $g(a) < \kappa$ there is a set T of compact elements with $a = \bigvee T$ and $|T| < \kappa$. The chain is directed, so for every $t \in T$, $t \leq a = \bigvee_\alpha a_\alpha$ gives an ordinal $\alpha(t) < \kappa$ with $t \leq a_{\alpha(t)}$. As κ is regular and $|T| < \kappa$, $\beta = \sup_{t\in T} \alpha(t) < \kappa$. Then

$$a = \bigvee T \leq a_\beta \leq a_\alpha \leq a$$

for all $\alpha \geq \beta$, i.e., $a_\alpha = a_\beta$ for all $\alpha \geq \beta$.

(ii)⇒(i). Let $a \neq 0$ and let C be the set of compact elements below a; since L is compactly generated, $a = \bigvee C$. Choose recursively, as long as

$$a_\xi := \bigvee_{\eta<\xi} t_\eta \neq a,$$

a compact element $t_\xi \leq a$ with $t_\xi \not\leq a_\xi$; this is possible, for otherwise C would lie below a_ξ and $a = \bigvee C \leq a_\xi$. If the recursion did not stop before κ, then $\{a_\xi : \xi < \kappa\}$ would be an ascending chain with $a_\xi < a_{\xi+1}$ for all ξ (because $t_\xi \leq a_{\xi+1}$, $t_\xi \not\leq a_\xi$), contradicting (ii). Hence $a_\xi = a$ for some $\xi < \kappa$ (necessarily $\xi \geq 1$), and $\{t_\eta : \eta < \xi\}$ is a generating set of a of cardinality $|\xi| < \kappa$.

The remaining assertions follow: (i) for κ implies (i) for every $\kappa' \geq \kappa$, and (i)⇔(ii) holds for each regular cardinal, so the two conditions have the same least regular solution. □

Remark 3.4. Since all the chains in a lattice L form a set, every lattice is λ-Noetherian for some regular cardinal λ. It also goes without saying that L is $\aleph_0$-Noetherian means that it is Noetherian in the usual sense.

Next, we present the counterparts of Remark 1.2 and Corollary 1.1 in [5].

Proposition 3.5. Let L be λ-Noetherian. Then for every $a \in L$, $[0, a]$ and $[a, 1]$ are μ-Noetherian for some $\mu \leq \lambda$. Conversely, if $[0, a]$ and $[a, 1]$ are λ_1-Noetherian and λ_2-Noetherian respectively, then L is $\max(\lambda_1, \lambda_2)$-Noetherian.

Proof . Let L be λ-Noetherian and let I be $[0, a]$ or $[a, 1]$. A well-ordered ascending chain of length λ in I is such a chain in L, so it is stable by Lemma 3.3. Thus (ii) of Lemma 3.3 holds for I with $\kappa = \lambda$, and by the same lemma I is μ-Noetherian for the least regular μ with this property; clearly $\mu \leq \lambda$.

Conversely, put $\lambda = \max(\lambda_1, \lambda_2)$ and let $\{a_\alpha : \alpha < \lambda\}$ be a well-ordered ascending chain in L. Then $\{a \wedge a_\alpha : \alpha < \lambda\}$ is an ascending chain in $[0, a]$ and $\{a \vee a_\alpha : \alpha < \lambda\}$ is an ascending chain in $[a, 1]$. Since $\lambda \geq \lambda_1, \lambda_2$, Lemma 3.3 shows that both chains are stable: there are $\beta, \gamma < \lambda$ with $a \wedge a_\alpha = a \wedge a_\beta$ for all $\alpha \geq \beta$ and $a \vee a_\alpha = a \vee a_\gamma$ for all $\alpha \geq \gamma$. Put $\delta = \max(\beta, \gamma) < \lambda$. For $\alpha \geq \delta$ we have $a_\delta \leq a_\alpha$, $a \wedge a_\delta = a \wedge a_\alpha$ and $a \vee a_\delta = a \vee a_\alpha$; hence $a_\alpha = a_\delta$ by (2.1). So every chain of length λ is stable. By Lemma 3.3, the Noetherian index λ_L of L satisfies $\lambda_L \leq \lambda$. On the other hand, by the first part, $\lambda_1, \lambda_2 \leq \lambda_L$, so $\lambda \leq \lambda_L$. Hence L is λ-Noetherian. □

Corollary 3.6. Let $a_1, a_2, \ldots, a_k$ be elements of L with each $[0, a_i]$ λ_i-Noetherian ($i = 1, 2, \ldots, k$). Then $[0, a_1 \vee a_2 \vee \cdots \vee a_k]$ is λ-Noetherian, where

$$\lambda \leq \max\{\lambda_1, \lambda_2, \ldots, \lambda_k\}.$$

Proof . We use induction on k; the case $k = 1$ is trivial. Put

$$b = a_1 \vee \cdots \vee a_{k-1}$$

and $c = a_k$. By induction $[0, b]$ is μ-Noetherian with

$$\mu \leq \max\{\lambda_1, \ldots, \lambda_{k-1}\}.$$

Moreover $[b, b \vee c] \cong [b \wedge c, c]$ is an interval of $[0, c]$, hence ν-Noetherian with $\nu \leq \lambda_k$ by Proposition 3.5. Applying the converse part of Proposition 3.5 to the lattice $[0, b \vee c]$ and the element b, we get that $[0, b \vee c]$ is $\max(\mu, \nu)$-Noetherian. □

For the counterpart of the following see [5], Proposition 1.2.

Proposition 3.7. Let λ be a regular cardinal. If a lattice L is such that for every $a \in L$ either $[0, a]$ or $[a, 1]$ is λ-Noetherian, then L is μ-Noetherian for some $\mu \leq \lambda^+$, where λ^+ is the least cardinal number greater than λ.

Proof . The cardinal λ^+ is regular, so by Lemma 3.3 it suffices to show that every ascending chain $\{a_\alpha : \alpha < \lambda^+\}$ in L is stable. Suppose that some chain is not stable.

Step 1: a strictly ascending subchain of length λ^+. Put $\alpha_0 = 0$; if α_ξ is defined choose $\alpha_{\xi+1} > \alpha_\xi$ with $a_{\alpha_{\xi+1}} \neq a_{\alpha_\xi}$ (possible since the chain is not stable); for a limit ordinal $\xi < \lambda^+$ put $\alpha_\xi = \sup_{\eta<\xi} \alpha_\eta$, which is $< \lambda^+$ by regularity of λ^+. Then $b_\xi = a_{\alpha_\xi}$ $(\xi < \lambda^+)$ is a strictly ascending chain.

Step 2. Put $a = b_\lambda$; then $a \neq 1$ because $b_\lambda < b_{\lambda+1}$. If $[0, a]$ is λ-Noetherian, then $\{b_\xi : \xi < \lambda\}$ is a strictly ascending chain of length λ in $[0, a]$, which contradicts Lemma 3.3. If $[a, 1]$ is λ-Noetherian, then $\{b_{\lambda+\xi} : \xi < \lambda\}$ is a strictly ascending chain of length exactly λ in $[a, 1]$ (note that, as cardinals, $\lambda + \xi = \lambda < \lambda^+$ for $\xi < \lambda$), and again Lemma 3.3 gives a contradiction. Hence every chain of length λ^+ is stable, and L is μ-Noetherian for the least regular μ with this property; clearly $\mu \leq \lambda^+$. □

Note that the hypothesis in Proposition 3.7 is used only for elements $a \neq 1$.

Corollary 3.8. If $[0, a]$ is λ-Noetherian for all $1 \neq a \in L$, then L is either λ-Noetherian or λ^+-Noetherian. In particular, if each $[0, a]$ is Noetherian for all $1 \neq a \in L$, then L is either Noetherian or $\aleph_1$-Noetherian, where $\aleph_1$ is the least uncountable cardinal.

Proof . By Proposition 3.7, L is μ-Noetherian for some regular $\mu \leq \lambda^+$. By Proposition 3.5, $\lambda \leq \mu$ (the intervals $[0, a]$ are λ-Noetherian). Hence $\mu \in \{\lambda, \lambda^+\}$. □

Example 3.9. Consider the set $\mathbb{N}_0$ of nonnegative integers ordered by division ($m \leq n$ if and only if $m \mid n$); then $\mathbb{N}_0$ is a lattice in which $m \vee n = \operatorname{lcm}(m, n)$ and $m \wedge n = \gcd(m, n)$. First we note that here the top element is 0 and the bottom element is 1.
a) $\mathbb{N}_0$ is a complete modular lattice.
b) $\mathbb{N}_0$ is not upper-continuous: take $X = \{2, 2^2, 2^3, \ldots\}$ and let a be an odd integer greater than 1. Then $\bigvee X = 0$ and

$$a \wedge (\bigvee_{x \in X} x) = a \wedge 0 = a,$$

while $a \wedge x = 1$ for all $x \in X$ and $\bigvee_{x \in X} (a \wedge x) = 1$; therefore

$$a \wedge (\bigvee_{x \in X} x) \neq \bigvee_{x \in X} (a \wedge x).$$

c) Since for all $k \geq 1$ the set consisting of k distinct primes is join-independent, the set of all primes is a join-independent set, and we conclude that $\operatorname{Gdim} \mathbb{N}_0 = \aleph_0$.
d) $(\mathbb{N}_0, \vee, \wedge)$ is Artinian and hence K-dim $(\mathbb{N}_0) = 0$.
e) It is routine to check that $(\mathbb{N}_0, \vee, \wedge)$ is not Noetherian (e.g., $1 \mid 2 \mid 4 \mid \cdots$), but for any $a \neq 0$, $[1, a]$ is finite. Moreover, a strictly ascending well-ordered chain in $\mathbb{N}_0$ has order type at most $\omega + 1$ (a nonzero integer has finitely many divisors, and 0 is the top element), so every chain of length $\aleph_1$ is stable. Thus $\mathbb{N}_0$ is not Noetherian but it is $\aleph_1$-Noetherian in the sense of chains (Lemma 3.3(ii)), in accordance with Corollary 3.8.
(b), (c), and (d) show that we cannot omit the upper-continuity in Theorem 2.10.

Lemma 3.10. Let L be a lattice. If $a, b \in L$ and $a \leq b$, then

$$s_\alpha([0, a]) = s_\alpha([0, b]) \wedge a$$

for all ordinals α.

Proof . The lemma holds trivially if $\alpha = 0$, so assume it holds for each ordinal $\gamma < \alpha$ and proceed by induction. For the sake of brevity, denote $s_\beta([0, a])$ by a_β and $s_\beta([0, b])$ by b_β for each ordinal $\beta \leq \alpha$. If α is a limit ordinal, then

$$a_\alpha = \bigvee_{\beta<\alpha} a_\beta = \bigvee_{\beta<\alpha} (b_\beta \wedge a) = \left(\bigvee_{\beta<\alpha} b_\beta \right) \wedge a = b_\alpha \wedge a.$$

If $\alpha = \beta + 1$, then

$$[a_\beta, b_\alpha \wedge a] = [b_\beta \wedge a, b_\alpha \wedge a] \cong [b_\beta, b_\beta \vee (b_\alpha \wedge a)] \subseteq [b_\beta, b_\alpha].$$

Since $[b_\beta, b_\alpha]$ is a semiatomic lattice, so is $[a_\beta, b_\alpha \wedge a]$, and consequently $b_\alpha \wedge a \leq a_\alpha$. On the other hand,

$$[b_\beta, b_\beta \vee a_\alpha] \cong [a_\alpha \wedge b_\beta, a_\alpha] \subseteq [a_\beta, a_\alpha],$$

because $a_\beta \leq b_\beta$ by the induction hypothesis. It follows that $[b_\beta, b_\beta \vee a_\alpha]$ is a semiatomic lattice, hence $b_\beta \vee a_\alpha \leq b_\alpha$ and so $a_\alpha \leq b_\alpha \wedge a$. Therefore, $a_\alpha = b_\alpha \wedge a$, i.e.,

$$s_\alpha([0, a]) = s_\alpha([0, b]) \wedge a.$$

□

Corollary 3.11. If L is a Loewy lattice, then for any $a \in L$, $l[0, a] = \alpha$ if and only if $a \leq s_\alpha(L)$ and $a \not\leq s_\beta(L)$ for all $\beta < \alpha$.

Definition 3.12. Let L be a lattice. The thickness of L, denoted by thick L, is defined to be the least cardinal number which is greater than or equal to the Goldie dimension of all sublattices $[a, b]$ of L; see also [4] and [5].

Remark 3.13. If L has Krull dimension, then by Lemma 2.9 and Theorem 2.10 every interval of L has finite Goldie dimension, so $\text{thick}(L) \leq \aleph_0$. In particular, if L is either Artinian or Noetherian, then $\text{thick}(L) \leq \aleph_0$. It is clear that if A is an interval of L, then $\text{thick}(A) \leq \text{thick}(L)$.

Proposition 3.14. Let L be a lattice. Then $\text{thick}(L) = 1$ if and only if L is a chain.

Proof . Let $\text{thick}(L) = 1$. If L is not a chain, there exist $a, b \in L$ such that $a \not\leq b$ and $b \not\leq a$. Then $a \wedge b < a < a \vee b$ and $a \wedge b < b < a \vee b$, so $\{a, b\}$ is a join-independent set of cardinality 2 in the interval $[a \wedge b, a \vee b]$. By the definition of Goldie dimension, $\text{Gdim}[a \wedge b, a \vee b] \geq 2$, and hence $\text{thick}(L) \geq 2$, contradicting $\text{thick}(L) = 1$. The converse is clear. □

Example 3.15. Let A be an antichain and let $L = A \cup \{0, 1\}$ be the lattice obtained by adding a least element 0 and a greatest element 1; then $\text{thick}(L) = |A|$; see Figure 1.

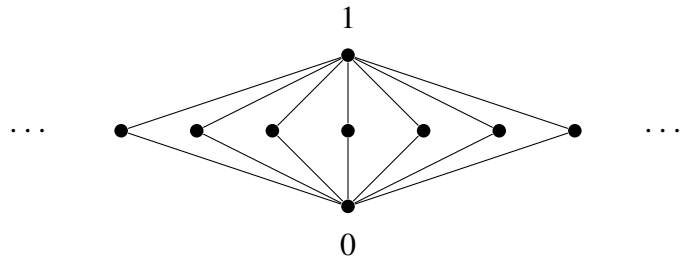


Figure 1: The complete lattice obtained from an antichain A (with $|A| = \aleph_0$ here) by adjoining a top element 1 and a bottom element 0: any two distinct middle elements are join-independent, so $\text{Gdim}\, L = |L|$ and, since $[0, 1] = L$ itself is the only nontrivial interval, $\text{thick}(L) = |A|$ as well.

Lemma 3.16. Let L be a lattice.

1. For all $a \in L$, $g(L) \leq g[0, a] + g[a, 1]$.
2. If L is semiatomic, then $g(L) \leq \text{Gdim}\, L$.

Proof . (1) For brevity, let $g[0,a] = \lambda$ and $g[a,1] = \mu$. There exists a set $S \subseteq [0,a]$ of compact elements with $|S| = \lambda$ and $a = \bigvee S$ (compactness in $[0,a]$ is the same as in L), and a set $T \subseteq [a,1]$ of elements compact in $[a,1]$ with $|T| = \mu$ and $1 = \bigvee T$. Let $t \in T$. As L is compactly generated, t is the directed join of the elements $a \vee F$, F a finite join of compact elements below t; since t is compact in $[a,1]$, $t = a \vee c_t$ for some such $F = c_t$, which is compact in L. Put

$$X = S \cup \{c_t : t \in T\}.$$

Then

$$\bigvee X = a \vee \bigvee_t c_t = \bigvee_t (a \vee c_t) = \bigvee T = 1,$$

and hence $g(L) \leq |X| \leq \lambda + \mu$.

(2) By Zorn's lemma, there is a maximal join-independent set P of atoms of L. If $e = \bigvee P \neq 1$, then, L being semiatomic, there is an atom $p \not\leq e$; then $p \wedge e = 0$ and, by modularity, $P \cup \{p\}$ is join-independent, contradicting maximality. So $1 = \bigvee P$. Atoms are compact, so P is a generating set of 1 and $g(L) \leq |P| \leq \operatorname{Gdim} L$. □

Lemma 3.17. If L is a Loewy lattice with $l(L) = m$ and $\operatorname{thick}(L) \leq n$, where m, n are positive integers, then $g(L) \leq mn$.

Proof . We proceed by induction on $l(L) = m$. If $m = 1$, then L is semiatomic and $\operatorname{Gdim} L \leq \operatorname{thick}(L) \leq n$; hence

$$g(L) \leq n = n \times 1$$

by Lemma 3.16(2). Suppose the statement is true for every Loewy lattice with Loewy length less than m, and let $l(L) = m > 1$. Put $c = s_{m-1}(L)$. By Corollary 3.11, $[0,c]$ is a Loewy lattice with $l[0,c] = m - 1$, and $\operatorname{thick}[0,c] \leq \operatorname{thick}(L) \leq n$ by Remark 3.13; so the induction hypothesis gives

$$g[0,c] \leq (m-1)n.$$

Since L is Loewy,

$$1 = s_m(L) = s([c,1]),$$

i.e., $[c, 1]$ is semiatomic, and $\mathrm{Gdim}[c, 1] \leq \mathrm{thick}(L) \leq n$; hence $g[c, 1] \leq n$ by Lemma 3.16(2). By Lemma 3.16(1),

$$g(L) \leq g[0, c] + g[c, 1] \leq (m - 1)n + n = mn.$$

□

Lemma 3.18. Let L be a Loewy lattice with $l(L) = \alpha$, and let κ be an infinite cardinal such that $|\alpha| \leq \kappa$ and $\mathrm{thick}(L) \leq \kappa$. Then

$$g(L) \leq \kappa.$$

In particular, if either α or $\mathrm{thick}(L)$ is infinite, then $g(L) \leq \max(|\alpha|, \mathrm{thick}(L))$.

Proof . Let $\kappa = \max(|\alpha|, \mathrm{thick}(L))$, an infinite cardinal. We show by transfinite induction on α the following statement: *every Loewy lattice L with $|l(L)| \leq \kappa$ and* $\mathrm{thick}(L) \leq \kappa$ *satisfies* $g(L) \leq \kappa$.

If $l(L) = 1$, then L is semiatomic and $g(L) \leq \mathrm{Gdim}\, L \leq \kappa$ by Lemma 3.16(2).

Let $l(L) = \beta + 1$ with $\beta \geq 1$ and put $c = s_\beta(L)$. By Corollary 3.11, $[0, c]$ is Loewy with $l[0, c] = \beta$ and $\mathrm{thick}[0, c] \leq \kappa$, so $g[0, c] \leq \kappa$ by the induction hypothesis. Also $[c, 1]$ is semiatomic with $\mathrm{Gdim}[c, 1] \leq \kappa$, hence $g[c, 1] \leq \kappa$. By Lemma 3.16(1), $g(L) \leq \kappa + \kappa = \kappa$.

Let $l(L) = \alpha$ be a limit ordinal. Then $1 = \bigvee_{0<\beta<\alpha} s_\beta(L)$. For $0 < \beta < \alpha$, since L is Loewy, Corollary 3.11 gives that $[0, s_\beta(L)]$ is a Loewy lattice with $l[0, s_\beta(L)] = \beta$, and $\mathrm{thick}[0, s_\beta(L)] \leq \kappa$ by Remark 3.13. Hence $[0, s_\beta(L)]$ has a generating set S_β of compact elements with $|S_\beta| \leq \kappa$. Then $\bigcup_\beta S_\beta$ generates 1 and

$$g(L) \leq \sum_{0<\beta<\alpha} |S_\beta| \leq |\alpha| \cdot \kappa = \kappa.$$

□

The following is the counterpart of the main result of [5].

Theorem 3.19. Let L be an Artinian lattice (recall that L is assumed to be compactly generated), and let λ be the least regular cardinal greater than $l(L)$. Then L is λ-Noetherian.

Proof . First, L is a Loewy lattice: if $c = s_{l(L)}(L) \neq 1$, then the nontrivial Artinian lattice $[c, 1]$ has an atom, whence $s_{l(L)+1}(L) > c$, contradicting the definition of $l(L)$. Let λ be as in the statement.

Step 1: no regular cardinal $\xi < \lambda$ has property (ii) of Lemma 3.3. Let ξ be a regular cardinal such that every ascending chain of length ξ in L is stable. The Loewy chain $\{s_\beta(L) : \beta < \xi\}$ is stable, so $s_\beta(L) = s_\mu(L)$ for all $\beta \geq \mu$, for some $\mu < \xi$. In particular, $s_\mu(L) = s_{\mu+1}(L)$, so $l(L) \leq \mu < \xi$. Hence $\xi \geq \lambda$.

Step 2: $g(a) < \lambda$ for all $a \in L$. Since L is Artinian, K-dim $L = 0$, so L has Krull dimension and thick$(L) \leq \aleph_0$ by Remark 3.13.

Case A: $l(L) = m < \omega$. Then $\lambda = \aleph_0$. For $0 \leq i < m$, the interval $M_i = [s_i(L), s_{i+1}(L)]$ is semiatomic and Artinian. By Theorem 2.10, Gdim $M_i < \infty$. Applying the proof of Lemma 3.16(2) inside M_i, the top element of M_i is a join of finitely many atoms; since M_i is Artinian, this implies that M_i has finite length. Thus each M_i has finite length, and since L is the union of the finite chain

$$0 = s_0(L) \leq s_1(L) \leq \cdots \leq s_m(L) = 1,$$

the lattice L itself has finite length. In particular, L is Noetherian, and $g(a) < \aleph_0 = \lambda$ for all $a \in L$ (alternatively, use Lemma 3.17).

Case B: $l(L)$ is infinite. Then $\kappa = |l(L)|$ is infinite, $\kappa < \lambda$, and $\aleph_0 \leq \kappa$. For $a \in L$, $[0, a]$ is a Loewy lattice with $l[0, a] \leq l(L)$ (Corollary 3.11) and thick$[0, a] \leq \aleph_0 \leq \kappa$. By Lemma 3.18 (with the infinite cardinal κ),

$$g(a) = g[0, a] \leq \kappa < \lambda.$$

By Step 2 and Lemma 3.3, (i) holds for λ, and by Step 1 no smaller regular cardinal satisfies (i). Hence L is λ-Noetherian. □

Example 3.20. Let p be a prime and let C_{p^∞} be the Prüfer p-group (quasicyclic p-group). Its subgroup lattice L_p is a complete, modular, upper-continuous, and compactly generated lattice. As a $\mathbb{Z}$-module, C_{p^∞} is Artinian but not Noetherian; hence L_p is an Artinian lattice which is not Noetherian. The compact elements of L_p are exactly the finite subgroups, and every element is a join of compact elements. The Loewy chain of L_p is

$$0 < C_p < C_{p^2} < \cdots < C_{p^n} < \cdots < C_{p^\infty},$$

so the Loewy length of L_p is $l(L_p) = \omega$. By Theorem 3.19, L_p is $\aleph_1$-Noetherian. However, L_p is not Noetherian, since the ascending chain

$$C_p < C_{p^2} < \cdots$$

does not stabilize. This example shows that the bound λ in Theorem 3.19 is sharp and cannot be improved to $\aleph_0$ in general.

The following is Lemma 1.9.1 in [7].

Lemma 3.21. If L is compactly generated, then $s(L)$ is the meet of all essential elements of L.

Now we are going to generalize results from [3].

Proposition 3.22. Let L be a compactly generated lattice. Then L has the descending chain condition on essential elements if and only if $[s(L), 1]$ is an Artinian lattice.

Proof . Let $[s(L), 1]$ be Artinian and

$$e_1 \geq e_2 \geq \cdots$$

be a descending chain of essential elements of L. By Lemma 3.21, $s(L) \leq e_i$ for all i, so

$$e_1 \geq e_2 \geq \cdots$$

is a chain in $[s(L), 1]$ and therefore, by assumption, there exists k such that $e_i = e_k$ for all $i \geq k$.

Conversely, assume that L has dcc on essential elements and let E be the (nonempty, as $1 \in E$) set of essential elements of L.

(a) E is closed under finite meets: if $e, f \in E$ and $x \neq 0$, then $e \wedge x \neq 0$ and so

$$(e \wedge f) \wedge x = f \wedge (e \wedge x) \neq 0.$$

(b) By dcc, E has a minimal element e_0; for any $e \in E$, $e_0 \wedge e \in E$ and $e_0 \wedge e \leq e_0$, so $e_0 \wedge e = e_0$, i.e., $e_0 \leq e$. Thus e_0 is the least element of E, and

$$e_0 = \bigwedge E = s(L)$$

by Lemma 3.21. Hence $s(L)$ is essential in L.

(c) Consequently, every $e \geq s(L)$ is essential (if $x \neq 0$ then $e \wedge x \geq s(L) \wedge x \neq 0$), so $[s(L), 1] \subseteq E$, and every descending chain in $[s(L), 1]$ is a descending chain of essential elements, hence stabilizes. So $[s(L), 1]$ is Artinian.

□

Proposition 3.23. If a is an essential element of a lattice L, such that $[0, a]$ has dcc on essential elements and $[a, 1]$ is Artinian, then L has dcc on essential elements.

Proof . Let

$$e_1 \geq e_2 \geq \cdots$$

be a descending chain of essential elements in L. Then

$$e_1 \wedge a \geq e_2 \wedge a \geq \cdots$$

is a descending chain of essential elements in $[0, a]$ (if $0 \neq x \leq a$, then $(e_i \wedge a) \wedge x = e_i \wedge x \neq 0$) and therefore there exists k such that $e_i \wedge a = e_k \wedge a$ for all $i \geq k$. On the other hand,

$$e_1 \vee a \geq e_2 \vee a \geq \cdots$$

is a descending chain of elements in $[a, 1]$, which is Artinian by assumption. Therefore, there exists l such that $e_i \vee a = e_l \vee a$ for all $i \geq l$. Now put $m = \max(k, l)$. For $i \geq m$, we have $e_i \leq e_m$, $e_i \wedge a = e_m \wedge a$, and $e_i \vee a = e_m \vee a$, so $e_i = e_m$ by (2.1), because L is modular. □

4 Comparison with related lattice-theoretic results

Our results are closely related to several earlier works on finiteness conditions for modular lattices. We briefly compare the main contributions.

Albu's lattice-theoretic Hopkins–Levitzki theorem [1]. Albu proved that every Artinian lattice of finite length over an upper-continuous modular lattice is Noetherian, and used this to give a noncategorical proof of the relative Hopkins–Levitzki theorem [1]. This was later extended by Albu and Smith using localization techniques for modular lattices [2]. The key difference from our approach is that these results require the lattice to have *finite length*, whereas our Theorem 3.19 applies to Artinian lattices with arbitrary Loewy length, including the infinite case (see Example 3.20). Moreover, while Albu's theorem is qualitative (Artinian ⇒ Noetherian in the finite-length case), our theorem provides the *explicit cardinal* λ relative to which the lattice is λ-Noetherian.

Karamzadeh–Motamedi's module-theoretic result [5]. The main theorem of [5] states that an Artinian module M is α-Noetherian if and only if α is the least regular cardinal greater than the supremum of the Loewy lengths of cyclic submodules of M. Our Lemma 3.3, Proposition 3.5, Proposition 3.7, Lemma 3.18, and Theorem 3.19 are lattice-theoretic counterparts of the results in [5]. The main technical difference is that the notion of a "cyclic submodule" has no direct lattice analogue, and we replace it by the invariants $\mathrm{thick}(L)$ and $g[0, a]$.

Albu–Iosif–Teply's QFD lattices [2]. A modular lattice L is QFD if every interval $[x, 1]$ has finite Goldie dimension [2]. Under the hypothesis $\text{thick}(L) \leq \aleph_0$, our lattice L is QFD (since $\text{thick}(L)$ bounds the Goldie dimension of all intervals $[a, b]$). Thus Theorem 3.19 can be viewed as an application of the QFD framework to the determination of a Noetherian-type cardinal for Artinian lattices. Conversely, our Lemma 3.16 provides a useful tool for estimating $g(L)$ in terms of interval invariants, which complements the Condition (C) characterization of QFD lattices in [2].

Năstăsescu–Van Oystaeyen's dimension theory [7]. The framework of Krull dimension and Gabriel dimension for lattices developed in [7] provides the foundational tools for our study. In particular, the fact that a lattice with Krull dimension has finite Goldie dimension (Theorem 2.10) ensures that $\text{thick}(L) \leq \aleph_0$ in the Artinian case (Remark 3.13), which is the key input for the application of Lemma 3.18 in the proof of Theorem 3.19.

5 Applications to rings and modules

Let R be a ring (not necessarily commutative) and let M be a left R-module. The lattice $\mathcal{L}(M)$ of submodules of M is complete, modular, upper-continuous, and compactly generated; its compact elements are exactly the finitely generated submodules. Consequently, all results of Section 3 apply to $\mathcal{L}(M)$.

Corollary 5.1. Let M be an Artinian left R-module, and let λ be the least regular cardinal greater than the Loewy length $l(M)$ of M. Then every submodule of M is generated by fewer than λ elements; equivalently, M is λ-Noetherian.

Proof . Apply Theorem 3.19 to the submodule lattice $\mathcal{L}(M)$. For a submodule N of M, the invariant $g(N)$ computed in $\mathcal{L}(M)$ coincides with the minimal cardinality of a generating set of N as an R-module. Hence $g(N) < \lambda$ for every $N \leq M$, which is precisely the assertion that M is λ-Noetherian. □

Corollary 5.1 recovers the main result of [5] in the Artinian case.

Corollary 5.2. Let M be a left R-module. Then M has the descending chain condition on essential submodules if and only if $M/\operatorname{Soc}(M)$ is an Artinian module.

Proof . Apply Proposition 3.22 to $\mathcal{L}(M)$. The socle $s(\mathcal{L}(M))$ is precisely $\operatorname{Soc}(M)$, and the interval $[s(\mathcal{L}(M)), 1]$ is isomorphic to the submodule lattice of $M/\operatorname{Soc}(M)$. Thus $[s(\mathcal{L}(M)), 1]$ is Artinian if and only if $M/\operatorname{Soc}(M)$ is Artinian. □

Taking $M = R$ regarded as a left module over itself in Corollary 5.2, we obtain the following theorem of Armendariz [3].

Corollary 5.3. A ring R has the descending chain condition on essential left ideals if and only if $R/\operatorname{Soc}({}_RR)$ is a left Artinian ring.

Remark 5.4. The same arguments apply to the lattice of subobjects of an object X in a Grothendieck category, provided that this lattice is compactly generated (which holds, for instance, when X is finitely generated and the category is locally finitely generated). In particular, the results of this paper yield corresponding statements for Artinian objects in such categories. Moreover, if M is a QFD module in the sense of [2], then $\operatorname{thick}(\mathcal{L}(M)) \leq \aleph_0$, and the estimates of Lemma 3.18 and Theorem 3.19 apply.

6 Conclusion

In this paper, we have studied the notion of λ-Noetherian lattices, where λ is a regular cardinal, and determined the deviation of a compactly generated Artinian lattice from being Noetherian in terms of its Loewy length. The main result (Theorem 3.19) states that such a lattice L is λ-Noetherian, where λ is the least regular cardinal greater than the Loewy length $l(L)$ of L. This extends the module-theoretic theorem of Karamzadeh and Motamedi [5] to the lattice setting.

Along the way, we developed the basic theory of λ-Noetherian lattices and related the invariants Gdim, thick, and g to the Loewy length. As an application, we obtained a lattice-theoretic generalization of a theorem of Armendariz [3] on rings with the descending chain condition on essential left ideals (Propositions 3.22 and 3.23), and we derived explicit consequences for modules and rings (Corollaries 5.1–5.3). The Prüfer p-group example (Example 3.20) shows that the bound λ in Theorem 3.19 is sharp and cannot be improved to $\aleph_0$ in general.

Several questions remain open. For instance, it would be of interest to determine whether the hypothesis of compact generation can be relaxed, or to what extent the results extend to non-modular lattices. Another natural direction is to investigate the relationship between the Loewy length and other cardinal invariants of lattices, such as the Gabriel dimension or the Noetherian dimension. We leave these problems for future work.

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Conflict of interest
The authors declare that they have no conflict of interest.

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