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© **Research paper**

A study of expansive automorphisms in a pointfree topology

Ali Akbar Estaji [a], Ali Barzanouni [a,*]

[a] *Faculty of Mathematics and Computer Science, Hakim Sabzevari University, Sabzevar, Iran*

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Abstract

We introduce and investigate expansive automorphisms in the setting of frames and commutative rings, providing a pointfree extension of classical expansivity in topological dynamics. We define the notion of *uniform refinement expansive automorphisms* for frames and rings and establish their fundamental properties and characterizations in terms of covers, refinements, z-ideals, and radical ideals.

A dual correspondence between expansive frame isomorphisms and expansive homeomorphisms on their spectra is developed via functorial constructions, yielding a bridge between algebraic and topological dynamics. Furthermore, we prove finiteness results for positively expansive systems, showing that frames admitting such automorphisms have finite maximal spectra, with applications to compact spaces. For function rings $\mathcal{R}(L)$ and $C(X)$, we establish equivalences between algebraic expansivity and classical topological expansivity.

*Corresponding author
Email addresses: aaestaji@hsu.ac.ir (Ali Akbar Estaji), Alibarzanouni@yahoo.com (Ali Barzanouni)

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1 Introduction

Given a compact metric space $(X, dist)$, we say a homeomorphism $h : X \to X$ is expansive, if there is $\delta > 0$ such that if $x, y \in X$, $x \neq y$, then $dist(h^n(x), h^n(y)) > \delta$ for some $n \in \mathbb{Z}$. This idea, introduced by Utz in 1950 [28] and it studied in both topological and metric settings [7, 23, 8]. It is known that expansivity can be expressed independently of the metric. The key is to consider the action of h on the cover of X. In [2], the notion of expansivity is extended from topological dynamical systems to the setting of automorphisms of unital commutative rings.

In this paper we give pointfree analogue of expansivity for frame isomorphisms and ring automorphisms. Our primary contributions are as follows:

1. **Uniformly Refinement Expansive Automorphisms**: We introduce a pointfree analogue of expansivity for frame isomorphisms and ring automorphisms, defined in terms of covers and their refinements. This generalizes classical expansivity to algebraic settings.
2. **Duality and Functoriality**: We establish connections between expansive frame isomorphisms and expansive homeomorphisms on their spectra, leveraging the spectrum functor Σ and the open set functor $\mathfrak{O}$. This extends known results in topological dynamics to the pointfree realm.
3. **Finiteness Theorems**: For positively expansive systems, we prove finiteness results for the number of maximal elements in frames and the cardinality of spectra, generalizing analogous theorems for compact spaces [1].
4. **Ring-Theoretic Expansivity**: We define and characterize expansive automorphisms of commutative rings, particularly for function rings $C(X)$ and $\mathcal{R}(L)$, and explore their relationship with z-ideals and radical ideals [25, 14].

Our work builds on foundational results in pointfree topology [3, 6], dynamical systems [22], and ring theory [2], while introducing new techniques to study expansivity algebraically. The paper is organized as follows: Section 2 reviews frames, their spectra, and categorical preliminaries. Section 3 introduces expansive automorphisms for commutative rings, and Section 4 develops the analogous theory for frames. Section 5 establishes the duality between these notions, and Section 6 proves finiteness properties for positively expansive systems.

By bridging topological dynamics and algebraic structures, this work opens new avenues for exploring dynamical properties in pointfree settings and their algebraic counterparts. Our results also provide tools for analyzing the "chaotic" behavior of automorphisms in algebraic contexts, extending the reach of classical dynamical concepts.

2 Preliminary

We recall from [27] that a *frame* (or *locale*) is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \colon s \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. We denote by $\bot$ and $\top$ the bottom and top elements of L, respectively. A *frame homomorphism* (or *frame map*) is a map between frames that preserves finite meets, including the top element, and arbitrary joins, including the bottom element. An element $p \in L$ is said to be *prime* if $p < \top$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. The contravariant functor Σ, from the category **Frm** of frames to the category **Top** of topological spaces, assigns to each frame L its *spectrum* ΣL of prime elements, with

$$\Sigma_a = \{p \in \Sigma L \colon a \not\leq p\} \quad (a \in L)$$

as its open sets. Also, throughout this paper for every frame homomorphism $h \colon L \to M$, we define

$$\Sigma h \colon \Sigma M \to \Sigma L, \quad p \mapsto \bigvee \{a \in L \mid h(a) \leq p\}$$

According to [19, 27], $\Sigma \colon Frm \to Top$ does have the following properties:

- **Contravariant**: $\Sigma(g \circ h) = \Sigma h \circ \Sigma g$.
- **Identity-Preserving**: $\Sigma(\mathrm{id}_L) = \mathrm{id}_{\Sigma L}$.
- **Prime Preservation**: $\bigvee \{a \in L \mid h(a) \leq p\}$ is prime in L when p is prime in M.
- **Continuity**: $(\Sigma h)^{-1}(\Sigma_a) = \Sigma_{h(a)}$ for all $a \in L$.

Hence, $\Sigma :$ **Frm** $\longrightarrow$ **Top** given by $L \mapsto \Sigma L$ and $h \mapsto \Sigma h$ is a contravariant functor.

Let X be a topological space. The collection $\mathfrak{O}(X)$ of open subsets of X forms a **frame**. The **open set functor** $\mathfrak{O} \colon$ **Top** $\to$ **Frm** is contravariant which $\mathfrak{O}(f) := f^{-1} \colon \mathfrak{O}(Y) \to \mathfrak{O}(X)$ for every continuous function $f \colon X \to Y$.

Let L be a frame. The map $\eta_L : L \longrightarrow \mathfrak{O}\Sigma L$ given by $\eta_L(a) = \Sigma_a$ is a frame map (see [19, 27]). We use this notation repeatedly throughout the paper.

Similar to the definition of a cover in topological spaces, we define a **cover** of a complete lattice P is a subset $\mathcal{A} \subseteq P$ such that $\bigvee \mathcal{A} = \top$. The set of all covers of P is denoted $\mathrm{Cov}(P)$. For covers $\mathcal{A}, \mathcal{B} \in \mathrm{Cov}(P)$:

- $\mathcal{A}$ **refines** $\mathcal{B}$ (denoted $\mathcal{A} \prec_{\text{cov}} \mathcal{B}$) if:

  $$\forall a \in \mathcal{A},\ \exists b \in \mathcal{B} \text{ with } a \leq b$$

  Equivalently, $\mathcal{A} \subseteq \downarrow \mathcal{B}$, where $\downarrow \mathcal{B} = \{p \in P \mid p \leq b \text{ for some } b \in \mathcal{B}\}$.

- The refinement relation $\prec_{\text{cov}}$ is:
  - Reflexive and transitive
  - Not antisymmetric (unlike lattice order $\leq$)

The set of all finite covers of P is denoted $\text{Cov}_F(P)$. The set of finite covers $\text{Cov}_F(P)$ has:

- **Meets**: For $\mathcal{A}, \mathcal{B} \in \text{Cov}_F(P)$,

  $$\mathcal{A} \wedge \mathcal{B} = \{a \wedge b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$$

  is a finite cover refining both $\mathcal{A}$ and $\mathcal{B}$.

- **Joins**: $\mathcal{A} \vee \mathcal{B} = \mathcal{A} \cup \mathcal{B}$.

By analogy with compact topological spaces, we consider a complete lattice P is **compact** if every cover has a finite subcover:

$$\forall \mathcal{A} \in \text{Cov}(P),\ \exists \mathcal{A}_0 \in \text{Cov}_F(P) \text{ with } \mathcal{A}_0 \subseteq \mathcal{A}$$

Each element x in a frame L, we set $x^* := \bigvee \{y \in L \colon x \wedge y = \bot\}$. For $x, y \in L$, we say that y is **rather below** x (written as $y \prec x$) if $x^* \vee x = \top$. We have that $y \prec x$ if and only if there is $v \in L$ such that $y \wedge v = \bot$ and $v \vee x = \top$. We call L a **regular frame** if for each $x \in L$, $x = \bigvee \{y \in L \colon y \prec x\}$.

Recall from [6] (see also [3]) that the frame of reals $\mathcal{L}(\mathbb{R})$ is obtained by taking the ordered pairs (p, q) of rational numbers as generators and imposing the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$.

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$.

(R3) $(p, q) = \bigvee\{(r, s) : p < r < s < q\}$.

(R4) $\top = \bigvee\{(p, q) : p, q \in \mathbb{Q}\}$.

For the pairs $(p, q) \in \mathbb{Q}^2$, we let:

$$< p, q >:= \{x \in \mathbb{R} : p < x < q\}.$$

The set $\mathcal{R}(L)$ of all frame homomorphisms from $\mathcal{L}(\mathbb{R})$ to L has been studied as an f-ring in [6].

Corresponding to every operation $\diamond : \mathbb{Q}^2 \to \mathbb{Q}$ (in particular $+, ., \wedge, \vee$) we have an operation on $\mathcal{R}(L)$, denoted by the same symbol $\diamond$, defined by:

$$\alpha \diamond \beta(p, q) = \bigvee \{\alpha(r, s) \wedge \beta(u, w) :< r, s > \diamond < u, w > \subseteq < p, q >\},$$

where $< r, s > \diamond < u, w > \subseteq < p, q >$ means that for each $r < x < s$ and $u < y < w$ we have $p < x \diamond y < q$. For any $\alpha \in \mathcal{R}(L)$ and $p, q \in \mathbb{Q}$, $(-\alpha)(p, q) = \alpha(-q, -p)$ and for every $r \in \mathbb{R}$, define the constant frame map $\mathbf{r} \in \mathcal{R}(L)$ by $\mathbf{r}(p, q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p, q) = \bot$. An element α of $\mathcal{R}(L)$ is said to be bounded if there exists $n \in \mathbb{N}$ such that $\alpha(-n, n) = \top$. The set of all bounded elements of $\mathcal{R}(L)$ is denoted by $\mathcal{R}^*(L)$ which is a sub-f-ring of $\mathcal{R}(L)$.

The *cozero map* is the map $coz : \mathcal{R}(L) \to L$, defined by

$$\mathrm{coz}(\alpha) = \bigvee \{\alpha(p, 0) \vee \alpha(0, q) : p, q \in \mathbb{Q}\}.$$

A *cozero element* of L is an element of the form $\mathrm{coz}(\alpha)$ for some $\alpha \in \mathcal{R}(L)$ (see [6]). The cozero part of L, is denoted by $\mathrm{Coz}L$. It is well known that L is called **completely regular** if and only if $\mathrm{coz}(L)$ generates L. For $A \subseteq \mathcal{R}(L)$, we set $Coz[A] := \{\mathrm{coz}(\alpha) : \alpha \in A\}$ and for $A \subseteq \mathrm{Coz}L$, we write $\mathrm{Coz}^{\leftarrow}[A]$ to designate the family maps $\{\alpha \in \mathcal{R}(L) : \mathrm{coz}(\alpha) \in A\}$.

Let R be a commutative ring with identity. An ideal $I \subseteq R$ is a **radical ideal** if it satisfies $\sqrt{I} = I$, where the **radical** of I is defined as:

$$\sqrt{I} = \{r \in R \mid \exists n \geq 1, r^n \in I\}.$$

Equivalently, I is radical if:

$$\forall r \in R,\ (\exists n \geq 1, r^n \in I) \implies r \in I.$$

For a commutative ring R with $\mathrm{Id}(R)$ and $\mathrm{RId}(R)$ as its ideals and radical ideals rep.

The poset $(\mathrm{RId}(R), \subseteq)$ forms a complete lattice with:

- **Meets (infima)**: Given $\{I_j\}_{j \in J} \subseteq \mathrm{RId}(R)$,

$$\bigwedge_{j \in J} I_j = \bigcap_{j \in J} I_j.$$

- **Joins (suprema)**: For $\{I_j\}_{j \in J} \subseteq \mathrm{RId}(R)$,

$$\bigvee_{j \in J} I_j = \sqrt{\sum_{j \in J} I_j}.$$

- **Top element**: $\top = R$
- **Bottom element**: $\bot = \sqrt{(0)}$ (nilradical of R)

For any ideals I, J in a commutative ring R:

1. $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$
2. $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$

- A **Ideal cover** is a subset $\mathcal{A} \subseteq \mathrm{Id}(R)$ where: $\sum_{I \in \mathcal{A}} I = R$
- A **radical cover** is a subset $\mathcal{A} \subseteq \mathrm{RId}(R)$ where: $\sqrt{\sum_{I \in \mathcal{A}} I} = R$

Let

$$\mathrm{Max}(R) := \{M \in \mathrm{Id}(R) \colon M \text{ is a maximal ideal of } R\},$$

and

$$\mathscr{M}(a) := \{M \in \mathrm{Max}(R) \colon a \in M\}$$

for every $a \in R$.
An ideal I of a ring R is a *z-ideal* if whenever two elements of R are in the same set of maximal ideals and I contains one of the elements, then it also contains the other. We refer to z-ideals as defined in [25] as "z-ideals in the sense of Mason". We recall from [25] that every ideal I of R is contained in a least z-ideal, in fact, this z-ideal is the intersection of all z-ideals containing I and in the notation of Mason [25] also we denote it by I_z. For every ideal I of $\mathcal{R}(L)$, we have

$$I_z = \mathrm{Coz}^{\leftarrow}[\mathrm{Coz}[I]] = \{\alpha \in \mathcal{R}(L) \colon \mathrm{coz}\alpha \in \mathrm{Coz}[I]\}.$$

An ideal I of $\mathcal{R}(L)$ (or $C(X)$) is a z-ideal if and only if for any $\alpha \in \mathcal{R}(L)$ ($\alpha \in C(X)$) and any $\beta \in I$, $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ implies $\alpha \in I$ (for more details, see [10, 11, 12, 14, 20]).

3 Expansive Automorphisms of commutative Rings

In this section, we introduce and study expansive automorphisms of commutative rings, analogous to expansive homeomorphisms in topological dynamics.

Definition 3.1 (Expansive Automorphisms). Let R be a commutative ring with identity and $\alpha : R \to R$ a ring automorphism.

1. α is called **uniformly refinement expansive** if there exists a finite cover $\mathcal{A} \in \mathrm{Cov}\,(\mathrm{Id}(R))$ such that for every cover $\mathcal{B} \in \mathrm{Cov}\,(\mathrm{Id}(R))$, there exists $n \in \mathbb{N} \cup \{0\}$ satisfying
$$\bigwedge_{|i|\leq n} \alpha^{-i}(\mathcal{A}) \prec_{\mathrm{cov}} \mathcal{B}.$$
Such $\mathcal{A}$ is called a **uniform refinement expansive cover**.
2. α is called **radical uniformly refinement expansive** if the above condition holds with $\mathrm{Id}(R)$ replaced by $\mathrm{RId}(R)$.
3. α is called **z-uniformly refinement expansive** if the condition holds for covers of $\mathrm{ZId}(R)$.

The following lemma holds, and its proof follows directly from the definition of radical ideals and the fact that automorphisms preserve powers of elements.

Lemma 3.2. For any ring automorphism $\alpha : R \to R$ and ideal $I \subseteq R$,
$$\alpha^{-1}(\sqrt{I}) = \sqrt{\alpha^{-1}(I)}.$$

Proposition 3.3. Every uniformly refinement expansive automorphism is radical uniformly refinement expansive.

Proof . Let α be uniformly refinement expansive with expansive cover $\mathcal{A} \subseteq \mathrm{Id}(R)$. We set $\sqrt{\mathcal{A}} := \{\sqrt{I} \mid I \in \mathcal{A}\}$ and it is a finite cover of $\mathrm{RId}(R)$.

For any $\mathcal{B} \in \mathrm{Cov}(\mathrm{RId}(R))$, there exists $n \geq 0$ such that:
$$\bigwedge_{|i|\leq n} \alpha^{-i}(\mathcal{A}) \prec_{\mathrm{cov}} \mathcal{B}.$$
Using Lemma 3.2, we obtain:
$$\bigwedge_{|i|\leq n} \alpha^{-i}(\sqrt{\mathcal{A}}) = \sqrt{\bigwedge_{|i|\leq n} \alpha^{-i}(\mathcal{A})} \prec_{\mathrm{cov}} \sqrt{\mathcal{B}} = \mathcal{B}.$$
□

Recall that for an ideal I, we define:
$$M(I) = \{\mathfrak{m} \in \mathrm{Max}(R) \mid I \subseteq \mathfrak{m}\}.$$
When $\{\mathscr{M}(a) \mid a \in I\}$ is closed under finite intersections, the z-closure is given by:
$$I_z = \{a \in R \mid \exists b \in I, \mathscr{M}(b) \subseteq \mathscr{M}(a)\}.$$

Proposition 3.4. Let $\alpha : R \to R$ be an automorphism and assume $\{\mathscr{M}(a) \mid a \in I\}$ is closed under finite intersections for all ideals I. If α is radical uniformly refinement expansive, then it is z-uniformly refinement expansive.

Proof . Let $\mathcal{A}$ be a radical expansive cover. Then $\mathcal{A}_z := \{I_z \mid I \in \mathcal{A}\}$ is a cover of $\mathrm{ZId}(R)$.

For any cover $\mathcal{B} \subseteq \mathrm{ZId}(R)$, there exists $n \geq 0$ such that:

$$\bigwedge_{|i|\leq n} \alpha^{-i}(\mathcal{A}) \prec_{\mathrm{cov}} \mathcal{B}.$$

The key observation is:

$$\left(\bigwedge_{|i|\leq n} \alpha^{-i}(\mathcal{A})\right)_z = \bigwedge_{|i|\leq n} \alpha^{-i}(\mathcal{A}_z).$$

Thus, $\bigwedge_{|i|\leq n} \alpha^{-i}(\mathcal{A}_z) \prec_{\mathrm{cov}} \mathcal{B}$. □

The assumption $\{\mathscr{M}(a) \mid a \in I\}$ on closure under finite intersections is essential in above proposition; its satisfied in completely regular spaces/frames. This proposition essentially shows that radical expansivity implies z-expansivity, which is important for function rings.

Corollary 3.5.
1. For X a completely regular space, if $\alpha : C(X) \to C(X)$ is radical uniformly refinement expansive, then it is z-uniformly refinement expansive.
2. For L a completely regular frame, if $\alpha : \mathcal{R}(L) \to \mathcal{R}(L)$ is radical uniformly refinement expansive, then it is z-uniformly refinement expansive.

Proof . Both results follow from Proposition 3.4 since:

- In $C(X)$, $f^2 + g^2 \in M$ implies $f, g \in M$ for maximal ideals M (see [24, 2D]).
- The analogous property holds for $\mathcal{R}(L)$ by complete regularity.

This ensures the finite intersection condition on $\mathscr{M}(a)$ sets is satisfied. □

4 Uniformly Refinement Expansive Frame Isomorphisms

In this section, we investigate expansive properties of frame isomorphisms, drawing parallels with expansive homeomorphisms in topological dynamics.

For any topological space X, we denote by $\mathfrak{O}(X)$ the frame of open subsets of X. Recall that for a continuous function $f\colon X_1 \to X_2$, the map $f^{-1}\colon \mathfrak{O}(X_2) \to \mathfrak{O}(X_1)$ is a frame

homomorphism. Dowker and Papert [9] established that when X_2 is Hausdorff, every frame homomorphism $\varphi\colon \mathfrak{O}(X_2) \to \mathfrak{O}(X_1)$ arises in this way from a unique continuous function.

Definition 4.1. Let L be a frame and $\alpha\colon L \to L$ a frame isomorphism.

1. α is called **uniformly refinement expansive** if there exists a finite cover $\mathcal{A}$ of L such that for every cover $\mathcal{B}$ of L, there exists $n \in \mathbb{N} \cup \{0\}$ satisfying:
$$\bigwedge_{|i|\leq n} \alpha^{-i}(\mathcal{A}) \prec_{\text{cov}} \mathcal{B}.$$
Such $\mathcal{A}$ is called a **uniform refinement expansive cover**.
2. α is called **positively uniformly refinement expansive** if there exists a finite cover $\mathcal{A}$ of L such that for every cover $\mathcal{B}$ of L, there exists $n \in \mathbb{N} \cup \{0\}$ satisfying:
$$\bigwedge_{i=0}^{n} \alpha^{-i}(\mathcal{A}) \prec_{\text{cov}} \mathcal{B}.$$

For notational convenience, given a frame isomorphism $\alpha\colon L \to L$ and a finite cover $\mathcal{A}$ of L, we define:

$$\mathcal{A}^{\pm}_{\alpha,n} := \bigwedge_{|i|\leq n} \alpha^{-i}(\mathcal{A}) \text{ and}$$
$$\mathcal{A}^{+}_{\alpha,n} := \bigwedge_{i=0}^{n} \alpha^{-i}(\mathcal{A})$$

for every $n \in \mathbb{N} \cup \{0\}$.

The two notions - uniform refinement expansivity and positive uniform refinement expansivity - are analogues of two-sided and one-sided expansivity in classical dynamical systems. The distinction lies in the index set used in the refinement condition ($|i| \leq n$ versus $0 \leq i \leq n$). Moreover, $\mathcal{A}^{\pm}\alpha, n$ always refines $\mathcal{A}^{+}\alpha, n$, whereas the converse does not necessarily hold.

The following results establish connections between expansive frame isomorphisms and expansive homeomorphisms of topological spaces.

Lemma 4.2. Let $\alpha\colon L \to L$ be a frame isomorphism. Then:

1. For every $(n, a) \in \mathbb{Z} \times L$, $\eta(\alpha^n(a)) = (\Sigma\alpha)^{-n}(\eta(a))$.
2. If $\mathcal{A}$ is a cover of L, then for every $n \in \mathbb{N} \cup \{0\}$:
$$(\eta(\mathcal{A}))^{\pm}_{\Sigma\alpha,n} = \eta(\mathcal{A}^{\pm}_{\alpha,n}).$$

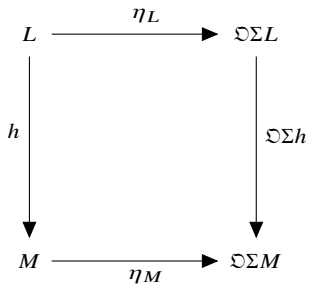
We recall from [27, 1.6] that a frame homomorphism $h\colon M \to L$ is co-dense if

$$h(a) = \top \Rightarrow a = \top.$$

In order to make the reader more familiar with frame map η_L, we recall from [19] that if $h : L \longrightarrow M$ is a frame map and $a \in L$, then

$$(\mathfrak{O}\Sigma h)(\eta_L)(a) = (\mathfrak{O}\Sigma h)(\Sigma_a) = (\Sigma h)^{-1}(\Sigma_a) = \Sigma_{h(a)} = \eta_M\big(h(a)\big).$$

Therefore, the following diagram is commutative.



Proposition 4.3. Let $\alpha\colon L \to L$ be a frame isomorphism with η_L co-dense.

1. If α is uniformly refinement expansive with expansive cover $\mathcal{A}$, then $\Sigma\alpha$ is an expansive homeomorphism with expansive open cover $\eta_L(\mathcal{A})$.
2. If $\Sigma\alpha$ is an expansive homeomorphism with expansive open cover $\{\Sigma_{a_i}\}_{i=1}^m$ and η_L is injective, then α is uniformly refinement expansive with expansive cover $\{a_i\}_{i=1}^m$.

Proof . (1) Let $\mathcal{B} := \{\Sigma_a\}_{a\in V}$ be an open cover of ΣL. Since η_L is co-dense, V is a cover of L. By expansiveness of α, there exists $n \in \mathbb{N} \cup \{0\}$ such that $\mathcal{A}^{\pm}_{\alpha,n} \prec_{\text{cov}} V$. Consequently:

$$(\eta(\mathcal{A}))^{\pm}_{\Sigma\alpha,n} = \eta(\mathcal{A}^{\pm}_{\alpha,n}) \prec_{\text{cov}} \eta(V) = \mathcal{B}.$$

(2) Let $\mathcal{H} := \{\Sigma_{a_i}\}_{i=1}^m$ be an expansive open cover of $\Sigma\alpha$. Since η_L is co-dense, $\mathcal{A} := \{a_i\}_{i=1}^m$ is a cover of L. For any cover $\mathcal{B}$ of L, $\mathcal{K} := \{\Sigma_b\}_{b\in\mathcal{B}}$ is an open cover of $\Sigma\alpha$, so there exists $n \in \mathbb{N} \cup \{0\}$ such that:

$$\eta(\mathcal{A}^{\pm}_{\alpha,n}) = (\eta(\mathcal{A}))^{\pm}_{\Sigma\alpha,n} \prec_{\text{cov}} \eta(\mathcal{B}).$$

Injectivity of η implies $\mathcal{A}^{\pm}_{\alpha,n} \prec_{\text{cov}} \mathcal{B}$. □

The following proposition is a characterization of finiteness of L via the identity maps expansivity.

Proposition 4.4. Let L be a frame and $\alpha\colon L \to L$ a frame isomorphism.

1. Every positively uniformly refinement expansive isomorphism is uniformly refinement expansive.
2. If α is (positively) uniformly refinement expansive, then L is compact.
3. The identity isomorphism is positively uniformly refinement expansive if and only if L is finite.

Proof . (1) Immediate from the definitions.

(2) Let $\mathcal{A}$ be an expansive cover. For any cover $\mathcal{B}$, there exists n such that $\mathcal{A}^{\pm}_{\alpha,n} \prec_{\text{cov}} \mathcal{B}$. Since $\mathcal{A}^{\pm}_{\alpha,n}$ is finite, $\mathcal{B}$ has a finite subcover.

(3) The identity is positively expansive precisely when there exists a finite cover $\mathcal{A}$ that refines every cover of L, which occurs exactly when L is finite. □

Proposition 4.5. A frame isomorphism $\alpha\colon L \to L$ is uniformly refinement expansive if and only if α^n is uniformly refinement expansive for every $n \in \mathbb{Z} \setminus \{0\}$.

Proof . ($\Rightarrow$) Let $\mathcal{A}$ be an expansive cover for α. For fixed $n \in \mathbb{N}$, set $C := \mathcal{A}^{\pm}_{\alpha,n}$. For any cover $\mathcal{B}$, there exists n_0 such that $\mathcal{A}^{\pm}_{\alpha,n_0} \prec_{\text{cov}} \mathcal{B}$. Choose $t \in \mathbb{N}$ with $nt \geq n_0$. Then:

$$C^{\pm}_{\alpha^n,t} \prec_{\text{cov}} \mathcal{A}^{\pm}_{\alpha,nt} \prec_{\text{cov}} \mathcal{A}^{\pm}_{\alpha,n_0} \prec_{\text{cov}} \mathcal{B}.$$

The case for negative powers follows similarly.

($\Leftarrow$) Take $n = 1$. □

Proposition 4.6. A frame isomorphism $\alpha\colon L \to L$ is positively uniformly refinement expansive if and only if there exists a finite cover $\mathcal{A}$ such that for every cover $\mathcal{B}$, there exists $n \in \mathbb{N} \cup \{0\}$ with $\alpha^{-n}(\mathcal{A}) \prec_{\text{cov}} \mathcal{B}$.

Proof . ($\Rightarrow$) Let $\mathcal{A}$ be a positively expansive cover. There exists n such that $\mathcal{A}^{+}_{\alpha,n} \prec_{\text{cov}} \alpha(\mathcal{A})$, hence $\alpha^{-1}(\mathcal{A}^{+}_{\alpha,n}) \prec_{\text{cov}} \mathcal{A}$. By induction, $\alpha^{-k}(\mathcal{A}^{+}_{\alpha,n}) \prec_{\text{cov}} \mathcal{A}^{+}_{\alpha,n+k}$ for all $k \in \mathbb{N}$.

For any cover $\mathcal{B}$, there exists $m \geq n$ with $\mathcal{A}^{+}_{\alpha,m} \prec_{\text{cov}} \mathcal{B}$. Taking $k = m - n$ gives:

$$\alpha^{-k}(\mathcal{A}^{+}_{\alpha,n}) \prec_{\text{cov}} \mathcal{A}^{+}_{\alpha,m} \prec_{\text{cov}} \mathcal{B}.$$

($\Leftarrow$) If $\mathcal{A}$ satisfies the given condition, then for any $\mathcal{B}$, there exists n with $\mathcal{A}^{+}_{\alpha,n} \prec_{\text{cov}} \alpha^{-n}(\mathcal{A}) \prec_{\text{cov}} \mathcal{B}$. □

5 Expansive Automorphisms of $\mathcal{R}(L)$

In this section, we establish connections between expansive frame isomorphisms and expansive automorphisms of the ring $\mathcal{R}(L)$ of real-valued continuous functions on a frame L.

For any element $a \in L$, define the ideal:

$$M_a := \{\alpha \in \mathcal{R}(L) \mid \operatorname{coz}(\alpha) \leq a\}.$$

For a subset $A \subseteq L$, let:

$$M_A := \{M_a \mid a \in A\}.$$

The following proposition captures a precise correspondence between covers in the frame L and covers in the z-ideal lattice of $\mathcal{R}(L)$.

Proposition 5.1. Let L be a compact completely regular frame and $\alpha\colon L \to L$ a frame isomorphism. Then:

1. If $\mathcal{A}$ is a cover of L, then $M_{\mathcal{A}}$ is a cover of $\operatorname{ZId}(\mathcal{R}(L))$.
2. If $\mathcal{U} \in \operatorname{Cov}(\operatorname{ZId}(\mathcal{R}(L)))$, then:
   (a) $\{\operatorname{coz}(\alpha) \mid \alpha \in \bigcup \mathcal{U}\}$ is a cover of L.
   (b) $\{\bigvee_{\alpha\in I} \operatorname{coz}(\alpha) \mid I \in \mathcal{U}\}$ is a cover of L.
3. The correspondence preserves refinements:
   (a) If $\mathcal{A} \prec_{\text{cov}} \mathcal{B}$ in L, then $M_{\mathcal{A}} \prec_{\text{cov}} M_{\mathcal{B}}$ in $\operatorname{ZId}(\mathcal{R}(L))$.
   (b) If $\mathcal{U} \prec_{\text{cov}} \mathcal{V}$ in $\operatorname{ZId}(\mathcal{R}(L))$, then both:
   - $\{\operatorname{coz}(\alpha) \mid \alpha \in \bigcup \mathcal{U}\} \prec_{\text{cov}} \{\operatorname{coz}(\alpha) \mid \alpha \in \bigcup \mathcal{V}\}$.
   - $\{\bigvee_{\alpha\in I} \operatorname{coz}(\alpha) \mid I \in \mathcal{U}\} \prec_{\text{cov}} \{\bigvee_{\alpha\in I} \operatorname{coz}(\alpha) \mid I \in \mathcal{V}\}$.

Proof . (1) Since $\mathcal{A} = \{a_1, \ldots, a_k\}$ covers L and L is compact, there exist $\alpha_{i,j} \in M_{a_i}$ such that $\bigvee_{i,j} \operatorname{coz}(\alpha_{i,j}) = \top$. Thus $\sum_{i,j} \alpha_{i,j} = \mathbf{1}$, showing $M_{\mathcal{A}}$ covers $\mathcal{R}(L)$.

(2) For $\mathcal{U} = \{I_1, \ldots, I_k\}$, there exist $\alpha_i \in I_i$ with $\sum \alpha_i = \mathbf{1}$. Then:

$$\top = \operatorname{coz}(\mathbf{1}) = \operatorname{coz}\left(\sum \alpha_i\right) \leq \bigvee \operatorname{coz}(\alpha_i)$$

proving both cover conditions.

(3) The refinement properties follow directly from the definitions. □

Lemma 5.2. Let L be a compact completely regular frame and $\alpha\colon L \to L$ a frame isomorphism. Define $\theta\colon \mathcal{R}(L) \to \mathcal{R}(L)$ by $\theta(\gamma) = \alpha \circ \gamma$. Then:

1. θ is an f-isomorphism.
2. $\theta^n(M_a) = M_{\alpha^n(a)}$ for all $n \in \mathbb{Z}$ and $a \in L$.
3. For any cover $\mathcal{A}$ of L, $\theta(M_{\mathcal{A}}) = M_{\alpha(\mathcal{A})}$ is a z-cover.

Proof . (1) For $\beta, \gamma \in \mathcal{R}(L)$ and $\diamond \in \{+, \cdot, \wedge, \vee\}$:

$$\begin{aligned}\theta(\gamma \diamond \beta)(p,q) &= \alpha\left(\bigvee\{\gamma(r,s) \wedge \beta(u,w) \mid (r,s) \diamond (u,w) \leq (p,q)\}\right)\\ &= \bigvee\{\alpha \circ \gamma(r,s) \wedge \alpha \circ \beta(u,w) \mid (r,s) \diamond (u,w) \leq (p,q)\}\\ &= (\theta(\gamma) \diamond \theta(\beta))(p,q).\end{aligned}$$

(2) For $\beta \in \mathcal{R}(L)$:

$$\begin{aligned}\beta \in M_{\alpha^n(a)} &\iff \operatorname{coz}(\beta) \leq \alpha^n(a)\\ &\iff \alpha^{-n}(\operatorname{coz}(\beta)) \leq a\\ &\iff \operatorname{coz}(\theta^{-n}(\beta)) \leq a\\ &\iff \theta^{-n}(\beta) \in M_a\\ &\iff \beta \in \theta^n(M_a).\end{aligned}$$

(3) Immediate from (2). □

The following theorem is the main duality result for this section.

Theorem 5.3. Let L be a compact completely regular frame and $\alpha\colon L \to L$ a frame isomorphism. The following are equivalent:

1. α is (positively) uniformly refinement expansive.
2. The induced automorphism $\theta(\gamma) = \alpha \circ \gamma$ is (positively) z-uniformly refinement expansive for every $\gamma \in \mathcal{R}(L)$.

Proof . For the non-positive case: ($\Rightarrow$) Let $\mathcal{A}$ be an expansive cover for α. Given $\mathcal{U} \in \operatorname{Cov}(\operatorname{ZId}(\mathcal{R}(L)))$, Proposition 5.1 yields a cover $\mathcal{B} = \{\operatorname{coz}(\alpha) \mid \alpha \in \bigcup \mathcal{U}\}$ of L. There exists n such that $\mathcal{A}^{\pm}_{\alpha,n} \prec_{\text{cov}} \mathcal{B}$. Then:

$$(M_{\mathcal{A}})^{\pm}_{\theta,n} = M_{\mathcal{A}^{\pm}_{\alpha,n}} \prec_{\text{cov}} \mathcal{U}.$$

($\Leftarrow$) Let $\mathcal{U}$ be a z-expansive cover for θ with $\{\beta_1, \ldots, \beta_k\} \subseteq \bigcup \mathcal{U}$ summing to $\mathbf{1}$. For any cover $\mathcal{B}$ of L, $M_{\mathcal{B}}$ is a z-cover, so there exists n with $\mathcal{U}^{\pm}_{\theta,n} \prec_{\text{cov}} M_{\mathcal{B}}$. Then:

$$\{\operatorname{coz}(\beta_1), \ldots, \operatorname{coz}(\beta_k)\}^{\pm}_{\alpha,n} \prec_{\text{cov}} \mathcal{B}.$$

The positive case follows similarly. □

The next corollary establishes a connection between classical topological expansivity and its algebraic analogue.

Corollary 5.4. For a compact Tychonoff space X and homeomorphism $h\colon X \to X$, the following are equivalent:

1. h is (positively) expansive.
2. The induced automorphism $f \circ h$ of $C(X)$ is (positively) z-expansive for every $f \in C(X)$.

Proof . This follows from Theorem 5.3 via the isomorphism $C(X) \cong \mathcal{R}(\mathfrak{O}(X))$ and the fact that $\mathfrak{O}(X)$ is a compact completely regular frame. □

6 Finiteness Properties for Expansive Systems

This section is a strong example of algebraic-topological duality in action-algebraic finiteness criteria lead to topological finiteness results.

The next lemma provides a combinatorial bound on the number of maximal elements in terms of covers with bounded size refining arbitrary covers after finite meets.

Lemma 6.1. Let L be a frame. If there exists $r \in \mathbb{N}$ such that for every cover $\mathcal{A}$ of L, there is a cover $\mathcal{B}$ with:

1. $|\mathcal{B}| \leq r$ and
2. $\mathcal{B}^{\wedge k} \prec_{\text{cov}} \mathcal{A}$ for some $k \in \mathbb{N}$ (where $\mathcal{B}^{\wedge k}$ denotes the k-fold meet of $\mathcal{B}$ with itself),

then L has at most r maximal elements.

Proof . Suppose for contradiction that L has $r+1$ distinct maximal elements $m_1, \ldots, m_{r+1}$. For each $1 \leq i \leq r+1$, define:

$$t_i := \bigwedge_{\substack{j \neq i \\ 1 \leq j \leq r+1}} m_j.$$

Then $\mathcal{M} := \{t_1, \ldots, t_{r+1}\}$ is a cover of L with no proper subcovers, since:

$$\bigvee_{\substack{j \neq i \\ 1 \leq j \leq r+1}} t_j = m_i \neq 1 \quad \text{for each } i.$$

By hypothesis, there exists a cover $\mathcal{B}$ with $|\mathcal{B}| \leq r$ and $\mathcal{B}^{\wedge k} \prec_{\text{cov}} \mathcal{M}$ for some k. For each $b \in \mathcal{B}$, there exists $t_b \in \mathcal{M}$ with $b^{\wedge k} \leq t_b$, hence $b \leq t_b$. Thus $\{t_b \mid b \in \mathcal{B}\}$ is a proper subcover of $\mathcal{M}$, a contradiction. □

Theorem 6.2. Let L be a frame admitting a positively expansive isomorphism $\alpha : L \to L$. Then:

1. L has finitely many maximal elements.
2. If L is compact and completely regular, then ΣL is finite.
3. For a compact Tychonoff space X, if X admits a positively expansive homeomorphism, then X is finite.

Proof . (1) Let $\mathcal{A}$ be a positively expansive cover for α. Since $\alpha(\mathcal{A})$ is a cover, there exists $n \geq 0$ such that:

$$\mathcal{A}^+_{\alpha,n} \prec_{\text{cov}} \alpha(\mathcal{A}) \implies \alpha^{-1}(\mathcal{A}^+_{\alpha,n}) \prec_{\text{cov}} \mathcal{A}.$$

We prove by induction that for all $m \geq 0$:

$$\alpha^{-m}\left((\mathcal{A}^+_{\alpha,n})^{\wedge 2^m}\right) \prec_{\text{cov}} \mathcal{A}^+_{\alpha,n+m}.$$

The base case $m = 0$ is trivial. For the inductive step, applying α^{-1} to the mth case gives:

$$\alpha^{-(m+1)}\left((\mathcal{A}^+_{\alpha,n})^{\wedge 2^m}\right) \prec_{\text{cov}} \alpha^{-1}(\mathcal{A}^+_{\alpha,n+m}) \prec_{\text{cov}} \mathcal{A}^+_{\alpha,n+m+1}$$

where the last refinement follows from the initial observation. The $m + 1$ case follows by taking meets.

For any cover $\mathcal{B}$, there exists m such that:

$$(\alpha^{-m}(\mathcal{A}^+_{\alpha,n}))^{\wedge 2^m} \prec_{\text{cov}} \mathcal{B}.$$

Thus by Lemma 6.1, if $|\mathcal{A}^+_{\alpha,n}| = r$, then L has at most r maximal elements.

(2) By Lemma 5.2, $\theta(\gamma) = \alpha \circ \gamma$ is a positively z-expansive automorphism of $\mathcal{R}(L)$. The maximal ideals of $\mathcal{R}(L)$ correspond to points of ΣL, and by part (1) there are finitely many.

(3) This follows from (2) via the isomorphism $C(X) \cong \mathcal{R}(\mathfrak{O}(X))$ and the correspondence between points of X and maximal ideals of $C(X)$. □

Remark 6.3. The key technical tool is the exponential refinement property (the 2^m factor), which forces any frame with a positively expansive automorphism to have finite "topological complexity" as measured by its maximal elements.

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Conflict of interest
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